Real Analysis

03/05/2018

Last time:

- Box $B = I_1 \times \cdots \times I_d \rightsquigarrow m(B) = |I_1| \times \cdots \times |I_d|$
- Elementary sets $E = B_1 \cup B_2 \cup \cdots \cup B_n$
  - Can pick B_i's such that they are ~~disjoint~~ non-overlapping $\rightsquigarrow m(E) = m(B_1) + \cdots + m(B_n)$
- Jordan inner/outer measure $J_*(A) = \sup_{\mathcal{E} \ni E \subset A} m(E)$, $J^*(A) = \inf_{\mathcal{E} \ni F \supset A} m(F)$.
  - A bounded set $A \subset \mathbb{R}^d$ is Jordan measurable
    $\Leftrightarrow J_*(A) = J^*(A)$
    $\Leftrightarrow \forall \varepsilon > 0, \exists E \in \mathcal{E}$ s.t. $E \subset A$ and $J^*(A \setminus E) < \varepsilon$
    $\Leftrightarrow$ ....

    ($\leftrightarrow$ Riemann integral)
- Both the elementary measure $m: \mathcal{E} \to [0, +\infty)$ and the Jordan measure $J: \mathcal{J} \to [0, +\infty)$ satisfies (1) normalization : $m([0,1]^d) = 1$
  (2) finite additivity : $m(A \cup B) = m(A) + m(B)$ if $A \cap B = \phi$.
  (c) translation-invariance : $m(A + \{x\}) = m(A)$.

  Non-negativity + finite additivity $\Rightarrow$ monotonicity, subadditivity.

Today: Lebesgue outer measure

1. Lebesgue outer measure

- Some shortcome of Jordan measure.
  - NOT defined for any unbounded set.
    [We could allow to define the measure of $\mathbb{R}$ (or "reasonable" unbounded subsets) to be $+\infty$. However, it is NOT reasonable to define the measure of $\mathbb{Z}$ to be $+\infty$.]
  - Sets like $\mathbb{Q}$, $\mathbb{Z}$ etc are NOT Jordan measurable
  - There exists open sets and compact sets which are NOT Jordan measurable.

    For example, let's write $\mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, \cdots\}$

    Take a small $\varepsilon > 0$. Let
    $$A_\varepsilon = (q_1 - \varepsilon, q_1 + \varepsilon) \cup (q_2 - \tfrac{\varepsilon}{2}, q_2 + \tfrac{\varepsilon}{2}) \cup (q_3 - \tfrac{\varepsilon}{4}, q_3 + \tfrac{\varepsilon}{4}) \cup \cdots$$
    Then A_ε is open, ~~.......~~ bounded, but not Jordan measurable.

    why? check: $J_*(A_\varepsilon) \le 4\varepsilon$, $J^*(A_\varepsilon) = 1$

    Similarly $[-1, 2] \setminus A$ is compact, but not Jordan measurable.
  - The "limit" of Jordan measurable sets could be Jordan non-measurable.
    e.g. $Q_N = \{q_1, q_2, \cdots, q_N\} \to \mathbb{Q}$.

- Let's recall the Jordan outer measure for a bounded set

$$J^*(A) = \inf_{\mathcal{E} \ni F \supset A} m(F) = \inf \left\{ \sum_{i=1}^{k} |B_i| \;\middle|\; B_1, \cdots, B_k \text{ are boxes s.t. } A \subset \bigcup_{i=1}^{k} B_i \right\}.$$

Lebesgue's idea:

- allow the measure of large sets like $\mathbb{R}$, $\mathbb{R}^d - \mathbb{Z}^d$ etc. to be $+\infty$.
- instead of covering A by finitely many boxes, allow countable covering

Def. The Lebesgue outer measure of ANY subset $A \subset \mathbb{R}^d$ is

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |B_i| \;\middle|\; B_1, B_2, \cdots \text{ are boxes s.t. } A \subset \bigcup_{i=1}^{\infty} B_i \right\}.$$

Example: $m^*(\mathbb{Q}) = 0$ since $\mathbb{Q} = \bigcup_{i=1}^{\infty} \{q_i\}$, and each $\{q_i\}$ is an interval of length 0.

[If you don't like "length zero intervals," you can use the intervals in the open set $A_\varepsilon = \bigcup_{i=1}^{\infty} (q_i - 2^{i-1}\varepsilon, q_i + 2^{i-1}\varepsilon)$ that we mentioned above, to conclude that

$$m^*(\mathbb{Q}) \le \sum_{i=1}^{\infty} 2^i \varepsilon = 4\varepsilon \quad , \forall \varepsilon$$

Taking limit $\varepsilon \to 0$, again we get $m^*(\mathbb{Q}) = 0$.]

Example: By the same proof, we see:

For any countable subset $A \subset \mathbb{R}^d$, we have $m^*(A) = 0$.

A simple observation:

$$m^*(A) \le J^*(A)$$

[given any finite covering $\{B_i : i=1, \cdots, n\}$ of A by boxes, one can alway convert it into a countable covering by adding empty sets.]

- Denote $\mathcal{P}(\mathbb{R}^d)$ = the set of all subsets of $\mathbb{R}^d$.

Then the Lebesgue outer measure, $m^* : \mathcal{P}(\mathbb{R}^d) \to [0, +\infty]$, satisfies the following properties, which are known as the outer measure axiom and will be used to define abstract outer measure for abstract sets.

Prop: The Lebesgue outer measure $m^* : \mathcal{P}(\mathbb{R}^d) \to [0, +\infty]$ satisfies

(1) (Empty set) $m^*(\phi) = 0$.

(2) (Monotonicity) If $A_1 \subset A_2 \subset \mathbb{R}^d$, then $m^*(A_1) \le m^*(A_2)$

(3) (Countable sub-additivity) For any countable family $A_1, A_2, \cdots$ of sets in $\mathbb{R}^d$, one has $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^*(A_n)$

Proof. (1) $m^*(\phi) \le J^*(\phi) = 0 \Rightarrow m^*(\phi) = 0$.

(2) Since $A_1 \subset A_2$, any covering of A_2 is automatically a covering of A_1.

So by definition, $m^*(A_1) \le m^*(A_2)$.

(3) [Idea: "give yourself an epsilon of room" trick]

Take any $\varepsilon > 0$.

Cover A_n by a countable family of boxes, $B_{n,1}, B_{n,2}, \cdots, B_{n,k}, \cdots$, such that

$$m^*(A_n) \ge \sum_{k=1}^{\infty} m(B_{n,k}) - 2^{-n}\varepsilon, \quad \forall n.$$

and such that $A_n \subset \bigcup_{k=1}^{\infty} B_{n,k}$.

Then the boxes $\{B_{n,k} : n=1,2,\cdots, k=1,2,\cdots\}$ is a countable covering of $\bigcup_{n=1}^{\infty} A_n$.

So by definition,

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n,k=1}^{\infty} m(B_{n,k}) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} m(B_{n,k}) \le \sum_{n=1}^{\infty} m^*(A_n) + \underbrace{\sum_{n=1}^{\infty} 2^{-n}\varepsilon}_{=\varepsilon}$$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^*(A_n). \qquad \square$$

Rmk. Countable subadditivity axiom + empty set axiom $\Rightarrow$ finite subadditivity axiom

$$m^*\left(\bigcup_{n=1}^{N} A_n\right) \le \sum_{n=1}^{N} m^*(A_n).$$

However, finite subadditivity does NOT imply countable subadditivity.
For example, the Jordan outer measure satisfies the finite subadditivity, but does not satisfy the countable subadditivity since $J^*(\mathbb{Q}) > \sum_{n=1}^{\infty} J^*(\{q_n\})$

[In particular, the Jordan outer measure will NOT be an outer measure in the abstract sense.]

Rmk. We have seen that the Jordan outer measure does NOT satisfy finite additivity.
Similarly the Lebesgue outer measure does NOT satisfy finite additivity
[Think about the Banach-Tarski paradox that we mentioned in lecture 1.]

However, we do have finite additivity or even countable additivity for Lebesgue outer measure, if we assume the sets encountered are nice and/or does not "entangle".

Another simple observation is the translation-invariance of Lebesgue outer measure.

Prop. $\| \; m^*(A + \{x\}) = m^*(A)$.

Proof. A countable family $\{B_n\}$ is a covering of A if and only the family $\{B_n + \{x\}\}$ is a covering of $A + \{x\}$. $\square$

- Next we compute the Lebesgue outer measure for elementary sets.

Prop. Let E be an elementary set. Then $m^*(E) = m(E)$ (elementary measure, $\sum_{i=1}^{n} |B_i|$ for non-overlapping covering of boxes)

Proof. • Since any elementary set is Jordan measurable, we have

$$m^*(E) \leq J^*(E) = m(E).$$

• To prove the converse, we need to apply the ε-trick twice.

Trick: To prove $A = B$ (numbers), it is enough to prove $A \geq B$ and $A \leq B$.

To prove $A = B$ (sets), it is enough to prove $A \subset B$ and $A \supset B$

[Idea: Since $m^*(Q) \not\geq J^*(Q)$, we need to use some property that Q does not have.
Here, we will use the Heine-Borel property of $\mathbb{R}$, i.e. any open covering of any compact set admits a finite subcovering.
We will apply one ε-trick to "shrink" "construction-boxes" of E to compact ones, and apply another ε-trick to "enlarge" covering boxes of E to open ones.]

– Suppose $E = \bigcup_{i=1}^{N} B_i$, where B_i's are ~~disjoint box~~ non-overlapping boxes.
Shrink each B_i to a smaller compact box $\tilde{B}_i$, such that $|\tilde{B}_i| \geq |B_i| - \frac{\varepsilon}{N}$.
Then the set $\tilde{E} = \bigcup_{i=1}^{N} \tilde{B}_i$ is compact, $\tilde{E} \subset E$ and $m(\tilde{E}) \geq m(E) - \varepsilon$.
[If some B_i already satisfies $|B_i| < \frac{\varepsilon}{N}$, e.g. $|B_i| = 0$, then take $\tilde{B}_i = \phi$]

– Now let $C_1, C_2, \cdots$ be any countable family of boxes s.t. $E \subset \bigcup_{i=1}^{\infty} C_i$, and $\sum_{i=1}^{\infty} |C_i| \leq m^*(E) + \varepsilon$.
Enlarge each C_i be a larger open box $\tilde{C}_i$, such that $|\tilde{C}_i| \leq |C_i| + 2^{-i} \cdot \varepsilon$

$$\Rightarrow \sum_{i=1}^{\infty} |\tilde{C}_i| \leq \sum_{i=1}^{\infty} |C_i| + 2\varepsilon$$

Then we have

$$\tilde{E} \subset E \subset \bigcup_{i=1}^{\infty} C_i \subset \bigcup_{i=1}^{\infty} \tilde{C}_i.$$

So the sets $\tilde{C}_1, \tilde{C}_2, \cdots$ is an open covering of the compact set $\tilde{E}$.
By compactness, $\exists k(1), \cdots, k(N)$ s.t. $\tilde{E} \subset \bigcup_{i=1}^{N} \tilde{C}_{k(i)}$.
It follows

$$m(E) \leq m(\tilde{E}) + \varepsilon \overset{\text{finite additivity of elementary measure}}{\leq} \sum_{i=1}^{N} |\tilde{C}_{k(i)}| + \varepsilon \leq \sum_{i=1}^{\infty} |\tilde{C}_i| + \varepsilon$$

$$\leq \sum_{i=1}^{\infty} |C_i| + 3\varepsilon \leq m^*(E) + 4\varepsilon.$$

Letting $\varepsilon \to 0$, we get $m(E) \leq m^*(E)$.
So $m(E) = m^*(E)$. □

2. Lebesgue outer measure via open sets

- **Def.** We say two boxes B_1 and B_2 are almost disjoint if their interior do not intersect.

e.g. $[0,1]$ and $[1,3]$ are almost disjoint.

Similarly we say $\{B_1, B_2, \cdots\}$ are almost disjoint if $\overset{\circ}{B}_i \cap \overset{\circ}{B}_j = \phi, \forall i \neq j$. "interior"

Obviously if $B_1, \cdots, B_N$ are almost disjoint, then we still have finite additivity

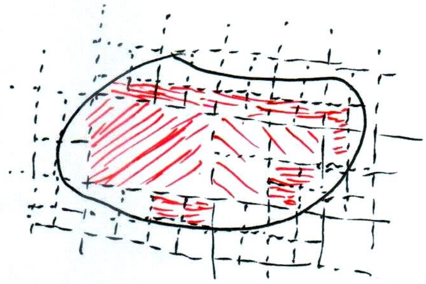
$$m(B_1 \cup \cdots \cup B_N) = \sum_{i=1}^{N} |B_i| .$$ boxes.

Thm. Any open set $A \subset \mathbb{R}^n$ can be expressed as a countable union of almost disjoint boxes. (which can be taken as closed boxes.)

Proof: Consider closed dyadic cube of the form

$$Q = \left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right] \times \cdots \times \left[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\right]$$

where $i_1, \cdots, i_d \in \mathbb{Z}$ and $n \in \mathbb{N}$.



Observation: Given any two different closed dyadic cubes, either they are almost disjoint, or one of them is totally contained in the other.

Let $\mathcal{A} = \{Q \mid Q \text{ is a closed dyadic cube}, Q \subset A\}$.

Then $\mathcal{A}$ is a countable family of closed boxes.

Claim: $A = \bigcup_{Q \in \mathcal{A}} Q$.

Proof of claim: Since each $Q \in \mathcal{A}$ is a subset of A, we have $\bigcup_{Q \in \mathcal{A}} Q \subset A$.

Conversely, for any $x \in A$, since A is open, by definition of openness, $\exists$ a small ball centered at x which is contained in A. Now take n large enough, one can find a closed dyadic box Q that lies in that small ball and contains x. So $x \in \bigcup_{Q \in \mathcal{A}} Q$ for any $x \in A$, i.e. $A \subset \bigcup_{Q \in \mathcal{A}} Q$.

It is possible that the dyadic boxes in $\mathcal{A}$ might intersect. However, according to the observation above, one can take

$$\mathcal{A}^* = \{Q \in \mathcal{A} \mid Q \text{ is NOT contained in any other dyadic box in } \mathcal{A}\}$$

Then $\mathcal{A}^*$ contains countable many almost disjoint closed boxes, and

$$A = \bigcup_{Q \in \mathcal{A}} Q = \bigcup_{Q \in \mathcal{A}^*} Q. \qquad \square$$

• **Lemma.** Let $A = \bigcup_{i=1}^{\infty} B_i$ be a countable union of almost disjoint boxes $B_1, B_2, \cdots$.
Then $m^*(A) = \sum_{i=1}^{\infty} |B_i|$

Proof. According to countable subadditivity,

$$m^*(A) \le \sum_{i=1}^{\infty} m^*(B_i) = \sum_{i=1}^{\infty} |B_i|.$$

Conversely, for each N, one has $B_1 \cup \cdots \cup B_N \subset A$. By monotonicity,

$$m^*(A) \ge m^*\Big(\bigcup_{i=1}^{N} B_i\Big) = m\Big(\bigcup_{i=1}^{N} B_i\Big) = \sum_{i=1}^{N} |B_i|.$$

Letting $N \to +\infty$, we get

$$m^*(A) \ge \sum_{i=1}^{\infty} |B_i|.$$

So $m^*(A) = \sum_{i=1}^{\infty} |B_i|$. $\square$

In particular, this gives us a way to compute the Lebesgue outer measure of any open set.

• **Prop.** (Outer regularity) For any $A \subset \mathbb{R}^d$, $m^*(A) = \inf\{m^*(U) : A \subset U,\ U \text{ is open}\}$

Proof. By monotonicity, $A \subset U \Rightarrow m^*(A) \le m^*(U)$.

So $m^*(A) \le \inf\{m^*(U) : A \subset U,\ U \text{ is open}\}$

To prove the converse, W.L.O.G, we assume $m^*(A) < \infty$.
By definition, $\exists$ boxes $B_1, B_2, \cdots$ s.t. $A \subset \bigcup_{i=1}^{\infty} B_i$ and

$$\sum_{i=1}^{\infty} |B_i| \le m^*(A) + \varepsilon.$$

Enlarge each B_i to an open box $\hat{B}_i \supset B_i$, s.t. $|\hat{B}_i| \le |B_i| + 2^{-i}\varepsilon$.
Then $\bigcup_{i=1}^{\infty} \hat{B}_i$ is an open set containing A, s.t.

$$m^*\Big(\bigcup_{i=1}^{\infty} \hat{B}_i\Big) \le \sum_{i=1}^{\infty} m^*(\hat{B}_i) = \sum_{i=1}^{\infty} |\hat{B}_i| \le \sum_{i=1}^{\infty} |B_i| + \varepsilon \le m^*(A) + 2\varepsilon.$$

(countable subadditivity)

It follows

$$\inf\{m^*(U) : A \subset U,\ U \text{ is open}\} \le m^*(A).$$

This completes the proof. $\square$