Real Analysis

Lecture 4
03/08/2018

Last time.

- Lebesgue outer measure $m^*: \mathcal{P}(\mathbb{R}^d) \to [0,+\infty]$.

$$m^*(A) = \inf\left\{ \sum_{i=1}^{\infty} |B_i| \;\middle|\; B_i\text{'s are boxes s.t. } A \subset \bigcup_{i=1}^{\infty} B_i \right\}$$
$$= \inf\{ m^*(U) \mid U \text{ is open, } A \subset U \}.$$

(U open $\Rightarrow \exists$ almost disjoint closed boxes $B_1, B_2, \cdots$ s.t. $U = \bigcup_{i=1}^{\infty} B_i$
$\Rightarrow m^*(U) = \sum_{i=1}^{\infty} |B_i|$.)

- Properties of Lebesgue outer measure
  - empty set $m^*(\phi) = 0$
  - monotonicity $A_1 \subset A_2 \Rightarrow m^*(A_1) \le m^*(A_2)$
  - subadditivity $m^*(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} m^*(A_i)$
- Examples of Lebesgue outer measure
  - $m^*(\text{countable set}) = 0$
  - $m^*(E) = m(E)$ if E is elementary

Two tricks in proof
- ε - trick
- "$=$" $\Leftrightarrow$ "$\ge$" and "$\le$".

Today: Lebesgue measure

1. Lebesgue measurable sets

- Recall: A bounded set A is Jordan measurable ~~[illegible]~~
$\Leftrightarrow \forall \varepsilon > 0, \exists F \in \mathcal{E}$ s.t. $A \subset F$ and $J^*(F \setminus A) < \varepsilon$.
- The Jordan outer measure $J^*(A) = \inf\{ m^*(F) \mid F \text{ is elementary, } A \subset F \}$.

Def: A set $A \subset \mathbb{R}^d$ is said to be <u>Lebesgue measurable</u> if $\forall \varepsilon > 0$, $\exists$ open set $U \subset \mathbb{R}^d$ s.t. $A \subset U$ and $m^*(U \setminus A) < \varepsilon$.

For Lebesgue measurable sets A, we will call
$$m(A) := m^*(A)$$
the <u>Lebesgue measure</u> of A. [It could be $+\infty$.]

Rmk: According to <u>this</u> definition, any measurable set is "nearly" open.
— Littlewood's 1st principle

Example:
- Any countable set is Lebesgue measurable, with Lebesgue measure 0.
[Reason: look at the example in last lecture, $\mathbb{Q} \rightsquigarrow \mathbb{Q}_\varepsilon$ open]
- In particular, ϕ and any finite set is Lebesgue measurable with measure 0.
- More generally, for any set A with $m^*(A) = 0$, it is always Lebesgue measurable.
[outer regularity]
- Any open set $U \subset \mathbb{R}^d$ is Lebesgue measurable. Any box is Lebesgue measurable.

Prop.: If $A_1, A_2, \cdots$ are Lebesgue measurable, then $\bigcup_{n=1}^{\infty} A_n$ is Lebesgue measurable.

Proof. (ε-trick 1) Fix any $\varepsilon > 0$. For any n, take an open set $U_n \supset A_n$ s.t.

$$m^*(U_n \setminus A_n) < 2^{-n} \cdot \varepsilon .$$

Then $U = \bigcup_{n=1}^{\infty} U_n$ is open, $U \supset \bigcup_{n=1}^{\infty} A_n$, and by subadditivity and monotonicity,

$$m^*\left(U \setminus \bigcup_{n=1}^{\infty} A_n\right) \le m^*\left(\bigcup_{n=1}^{\infty} (U_n \setminus A_n)\right) \le \sum_{n=1}^{\infty} m^*(U_n \setminus A_n) < \varepsilon .$$

So $\bigcup_{n=1}^{\infty} A_n$ is Lebesgue measurable. $\square$.

Prop.: Any compact set in $\mathbb{R}^d$ is Lebesgue measurable.

Proof: Let $A \subset \mathbb{R}^d$ be compact.

Take an open set $U \supset A$ s.t. $m^*(U) \le m^*(A) + \varepsilon$.

Since $U \setminus A$ is open, we can write $U \setminus A = \bigcup_{n=1}^{\infty} B_n$, where B_n's are almost disjoint closed boxes.

Let $C_m = \bigcup_{n=1}^{m} B_n$. Then C_m is a finite union of closed boxes. Thus C_m is closed.

Obviously $C_m \cap A = \phi$. Since A is compact, we have $d(A, C_m) > 0$.

According PSet 2, Part 1, Problem 3, we have

$$m^*(A) + m^*(C_m) = m^*(A \cup C_m) \le m^*(A \cup (U \setminus A)) = m^*(U) \le m^*(A) + \varepsilon .$$

Since A is compact, A is contained in a large box. So $m^*(A) < \infty$.

It follows

$$m^*(C_m) = m^*\left(\bigcup_{n=1}^{m} B_n\right) \overset{\text{(almost disjoint boxes)}}{=} \sum_{n=1}^{m} |B_n| < \varepsilon .$$

Let $m \to \infty$, we get

$$m^*(U \setminus A) = \sum_{n=1}^{\infty} |B_n| < \varepsilon .$$

So A is Lebesgue measurable. $\square$

Cor.: Any closed set in $\mathbb{R}^d$ is Lebesgue measurable.

Proof: Let A be closed. Let $A_n = A \cap \overline{B(0,n)}$. Then A_n is compact, and $A = \bigcup_{n=1}^{\infty} A_n$. So by the two propositions above, A is Lebesgue measurable. $\square$

Prop. If $A \subset \mathbb{R}^d$ is Lebesgue measurable, then $A^c = \mathbb{R}^d \setminus A$ is Lebesgue measurable.

Proof. Take open sets $U_n \supset A$ s.t. $m^*(U_n \setminus A) < \frac{1}{n}$.

Let $F_n = U_n^c$ and $F = \bigcup_{n=1}^{\infty} F_n$. Then F is Lebesgue measurable (why?)

Since $A^c \setminus F \subset A^c \setminus F_n$, and $m^*(A^c \setminus F_n) = m^*(U_n \setminus A) < \frac{1}{n}$, we get

$$m^*(A^c \setminus F) \le \lim_{n \to \infty} m^*(A^c \setminus F_n) = 0 .$$

So $A^c \setminus F$ is Lebesgue measurable.

As a consequence, $A^c = (A^c \setminus F) \cup F$ is Lebesgue measurable. $\square$

~~Cor: If A_1, A_2 are Lebesgue measurable, then $A_1 \setminus A_2$ is Lebes~~

Cor.: If $A_1, A_2, \cdots$ are Lebesgue measurable, then $\bigcap_{n=1}^{\infty} A_n$ is Lebesgue measurable.

Proof. $\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_n A_n^c\right)^c$. $\square$

Cor.: Any F_σ set is Lebesgue measurable. [F_σ = countable union of closed sets]
Any G_δ set is Lebesgue measurable. [G_δ = countable intersection of open sets.]

"$\sigma \longleftrightarrow$ countable union"

Cor.: The collection of all Lebesgue measurable subsets in $\mathbb{R}^d$ form a σ-algebra.

e.g.: A family of subsets $\mathcal{F}$ is called a σ-algebra if
(1) $\phi \in \mathcal{F}$
(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
(3) $A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

~~(2) Properties of Lebesgue measure~~

Cor.: If A_1, A_2 are Lebesgue measurable, then $A_1 \setminus A_2$ is Lebesgue measurable.

Proof: $A_1 \setminus A_2 = A_1 \cap A_2^c$. $\square$

We have used open sets to define Lebesgue measurability.
The next proposition says we can use closed sets to do the same task.

Prop.: $A \subset \mathbb{R}^d$ is Lebesgue measurable if and only if $\forall \varepsilon > 0$, $\exists$ closed set $F \subset A$ s.t.
$m^*(A \setminus F) < \varepsilon$.

Proof.
- Suppose A is Lebesgue measurable. Then A^c is Lebesgue measurable.
  So $\exists$ open set $U \supset A^c$ s.t. $m^*(U \setminus A^c) < \varepsilon$.
  Let $F = U^c$. Then $m^*(A \setminus F) = m^*(U \setminus A^c) < \varepsilon$.
- Suppose $\forall \varepsilon > 0$, $\exists$ closed set $F \subset A$ s.t. $m^*(A \setminus F) < \varepsilon$.
  Then $U = F^c$ is open, $U \supset A^c$, and $m^*(U \setminus A^c) = m^*(A \setminus F) < \varepsilon$.
  So A^c is Lebesgue measurable, i.e. A is Lebesgue measurable. $\square$

Crucial identity:
$A \setminus B = A \cap B^c$
So
$A \setminus U^c = A \cap U = U \setminus A^c$!

2. Additivity of Lebesgue measure

We have known that the Lebesgue outer measure don't satisfy additivity.
It turns out that if we restrict ourselves to Lebesgue measurable sets, then we do have (countable) additivity.

Thm.: Suppose $A_1, A_2, \cdots$ is a countable sequence of disjoint Lebesgue measurable sets. Then
$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

Rmk.: measure axioms: (1) empty set: $m(\phi) = 0$
(2) countable additivity.

Proof of countable additivity

[Trick: step by step, from special to general.]

→ Simple Case: Assume all A_n are compact.

If A_i, A_j are compact, and $A_i \cap A_j = \phi$, then $d(A_i, A_j) > 0$.

So by repeatly applying PSet 2, Part 1, Problem 3, we get

$$\sum_{n=1}^{N} m(A_n) = m\left(\bigcup_{n=1}^{N} A_n\right) \le m\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m(A_n).$$

Letting $N \to \infty$, we get $\sum_{n=1}^{\infty} m(A_n) = m\left(\bigcup_{n=1}^{\infty} A_n\right)$.

→ Middle Case: Assume all A_n's are bounded.

Fix any $\varepsilon > 0$.

Choose a closed set $F_n \subset A_n$ s.t. $m^*(A_n \setminus F_n) < \varepsilon \cdot 2^{-n}$. $\Rightarrow m(A_n) \le m(F_n) + \varepsilon \cdot 2^{-n}$.

Since F_n's are bounded, closed, they are disjoint compact sets.

Applying Simple Case to F_n, we get

$$\sum_{n=1}^{\infty} m(A_n) \le \sum_{n=1}^{\infty} m(F_n) + \varepsilon = m\left(\bigcup_{n=1}^{\infty} F_n\right) + \varepsilon \le m\left(\bigcup_{n=1}^{\infty} A_n\right) + \varepsilon \le \sum_{n=1}^{\infty} m(A_n) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get $\sum_{n=1}^{\infty} m(A_n) = m\left(\bigcup_{n=1}^{\infty} A_n\right)$.

→ General Case: Now only assume that A_n's are measurable.

Let $C_m = B(0,m) \setminus B(0,m-1)$. [The annulus centered at 0, with inner radius $m-1$ and outer radius m.]

Let $D_{m,n} = C_m \cap A_n$. Then $A_n = \bigcup_m D_{m,n}$, $\bigcup_n A_n = \bigcup_{m,n} D_{m,n}$.

Note: These $D_{m,n}$'s are all disjoint bounded Lebesgue measurable sets.

Applying Middle Case to $D_{m,n}$'s, we get

$$\sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m(D_{m,n}) = m\left(\bigcup_{n=1}^{\infty} A_n\right). \quad \square$$

Cor: Let $A_1 \subset A_2 \subset \cdots \subset \mathbb{R}^d$ be an increasing sequence of Lebesgue measurable sets. Then $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$.

[monotone convergence of measurable sets.]

Proof: Let $A_1' = A_1$, $A_n' = A_n \setminus A_{n-1}$.

Then $\bigcup_n A_n' = \bigcup_n A_n$, and A_n''s are disjoint, Lebesgue measurable, and (why?) $m(A_n') = m(A_n) - m(A_{n-1})$.

So

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{j=1}^{\infty} A_j'\right) = \sum_{j=1}^{\infty} m(A_j') = \lim_{n\to\infty} \sum_{j=1}^{n} m(A_j')$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left(m(A_j) - m(A_{j-1})\right) = \lim_{n\to\infty} m(A_n). \quad \square$$