Last time

- $A \subset \mathbb{R}^d$ is Lebesgue measurable $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists$ open set $U \supset A$ s.t. $m^*(U \setminus A) < \varepsilon$

  $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists$ closed set $F \subset A$ s.t. $m^*(A \setminus F) < \varepsilon$.

- $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$.

  $A_i \in \mathcal{L} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$, $\bigcap_{i=1}^{\infty} A_i \in \mathcal{L}$. $\leftarrow$ $\mathcal{L}$ is a "σ-algebra"

- Examples of measurable sets: open, closed, countable, F_σ, G_δ, Jordan measurable, ...
- $A_1 \subset A_2 \subset \cdots$, $A_i \in \mathcal{L}$ $\Rightarrow$ $m(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} m(A_i)$

  $A_1 \supset A_2 \supset \cdots$, $A_i \in \mathcal{L}$, $m(A_1) < \infty$ $\Rightarrow$ $m(\bigcap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} m(A_i)$

- Additivity. $A_i \in \mathcal{L}$ disjoint $\Rightarrow$ $m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$.

Today: Lebesgue measure (continued)

~~Carathéodory~~

1. Properties of Lebesgue measurable sets

- Let $A \subset \mathbb{R}^d$ be Lebesgue measurable. Then we have seen that as subsets in $\mathbb{R}^d$, A is "nice" in various aspects, e.g. it is "nearly open", "nearly close", etc. By "nearly", we do allow an "ε-error", i.e. a set with Lebesgue outer measure $\leq \varepsilon$. We can also ask for "better approximation", in the sense that the error has Lebesgue (outer) measure 0, with the loss that we can't require open/close sets (to approximate by).

Prop. $A \in \mathcal{L} \Leftrightarrow A = G \setminus N$, where G is a G_δ set, N is a null set

$\Leftrightarrow A = F \cup N$, where F is a F_σ set, ... — ...

Def. A set is a G_δ set if it is the countable intersection of open sets.
A set is a F_σ set — — — — — union — closed — ..
A set is a null set if its Lebesgue outer measure is 0.

Proof. Suppose $A \in \mathcal{L}$. Then $\exists$ open sets $U_n \supset A$ s.t. $m^*(U_n \setminus A) < \frac{1}{n}$.
Let $G = \bigcap_n U_n$. Then $A \subset G$. Denote $N = G \setminus A$, then $N \subset U_n \setminus A$, $\forall n$.
So $m^*(N) \leq m^*(U_n \setminus A) < \frac{1}{n}$. $\Rightarrow m^*(N) = 0$, i.e. N is a null set.

The converse is obvious: if $A = G \setminus N$, where G is a G_δ set (which is Lebesgue measurable) and N is a null set (which is also Lebesgue measurable), then A is Lebesgue measurable.

To prove $A \in \mathcal{L} \Leftrightarrow A = F \cup N$, where F is F_σ set and N is a null set, one can either do the same argument as above (i.e. pick $F_n \subset A$ closed s.t. $m^*(A \setminus F_n) < \frac{1}{n}$, then take $F = \bigcup_n F_n$ and $N = A \setminus F$), or one can use the fact $A \in \mathcal{L} \Leftrightarrow A^c \in \mathcal{L}$.

i.e.

$$A \in \mathcal{L} \Leftrightarrow A^c \in \mathcal{L} \Leftrightarrow A^c = \underset{\underset{G_\delta}{\uparrow}}{G} \setminus N \Leftrightarrow A = (G \setminus N)^c = (G \cap N^c)^c = \underset{\underset{F_\sigma}{\uparrow}}{G^c} \cup N. \qquad \square$$

· Another very nice property for Lebesgue measurable sets is

<u>Prop.</u> $A \in \mathcal{L} \Rightarrow \forall T \subset \mathbb{R}^d$, $m^*(T) = m^*(T \cap A) + m^*(T \setminus A)$.

<u>Proof</u>: · If T is open, then $T, T \cap A, T \setminus A \in \mathcal{L}$, and the conclusion follows from additivity. (if $m^*(T) = +\infty$, then $m^*(T) \le m^*(T \cap A) + m^*(T \setminus A) \Rightarrow m^*(T \cap A) + m^*(T \setminus A) = +\infty$.)

· For general T, if $m^*(T) < +\infty$, then we pick open set $U \supset T$ s.t. $m^*(U) \le m^*(T) + \varepsilon$. Then

$$m^*(T) \le m^*(T \cap A) + m^*(T \setminus A) \le m^*(U \cap A) + m^*(U \setminus A) = m^*(U) \le m^*(T) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$m^*(T) = m^*(T \cap A) + m^*(T \setminus A). \qquad \square$$

In fact, we have

<u>Thm</u>. (Caratheodory criterion) T.F.A.E.

(1) $A \in \mathcal{L}$

(2) For any box B, $|B| = m^*(B \cap A) + m^*(B \setminus A)$

(3) $\forall E \in \mathcal{E}$, $m(E) = m^*(E \cap A) + m^*(E \setminus A)$

(4) $\forall T \subset \mathbb{R}^d$, $m^*(T) = m^*(T \cap A) + m^*(T \setminus A)$

<u>Proof</u>: Obviously <u>(1) ⇒ (4) ⇒ (3) ⇒ (2)</u>

· <u>(2) ⇒ (3)</u>: Suppose $E = \bigcup_{i=1}^{n} B_n$ is a disjoint union of boxes. Then

$$m(E) \le m^*(E \cap A) + m^*(E \setminus A) \le \sum_{i=1}^{n} m^*(B_i \cap A) + \sum_{i=1}^{n} m^*(B_i \setminus A) = \sum_{i=1}^{n} |B_i| = m(E).$$

<u>(2)/(3) ⇒ (1)</u>: · First suppose $m^*(A) < \infty$.

Take disjoint boxes B_n s.t. $A \subset \bigcup_{n=1}^{\infty} B_n$ and $\sum_{n=1}^{\infty} |B_n| \le m^*(A) + \varepsilon$.

Enlarge each box B_n to open box $\tilde{B}_n$ s.t. $\tilde{B}_n \supset B_n$ and $|\tilde{B}_n| \le |B_n| + \varepsilon \cdot 2^{-n}$.

Let $U = \bigcup_n \tilde{B}_n$. Then $U \supset A$, U is open. Moreover,

$$m^*(A) + m^*(U \setminus A) \le \sum_n m^*(\tilde{B}_n \cap A) + \sum_n m^*(\tilde{B}_n \setminus A)$$
$$= \sum_n |\tilde{B}_n| \le \sum_n |B_n| + \varepsilon \le m^*(A) + 2\varepsilon.$$

Thus $m^*(U \setminus A) < 2\varepsilon$, i.e. $A \in \mathcal{L}$.

- Now consider the case $m^*(A) = +\infty$. Let B_n = the box centered at 0, with side length n.

Then $A = \bigcup_{n=1}^{\infty} (A \cap B_n)$. It remains to prove $A_n := A \cap B_n \in \mathcal{L}$.

$\Uparrow$ $\boxed{m^*(A_n) < \infty}$

Let B be any box, we need to check

$$|B| = m^*(B \cap A_n) + m^*(B \setminus A_n).$$

To prove this, we first observe

$$B \setminus (A \cap B_n) = B \cap (A \cap B_n)^c = B \cap (A^c \cup B_n^c) = (B \cap A^c) \cup (B \cap B_n^c)$$
$$= (B \cap A^c \cap B_n) \cup (B \cap A^c \cap B_n^c) \cup (B \cap B_n^c)$$
$$= (B \cap A^c \cap B_n) \cup (B \cap B_n^c)$$

Then $B \cap B_n$ is elementary, thus Lebesgue measurable

$$|B| \le m^*(B \cap A \cap B_n) + m^*(B \setminus (A \cap B_n)) \le m^*(B \cap A \cap B_n) + m^*(B \cap B_n \cap A^c) + m^*(B \cap B_n^c)$$
$$= m^*(B \cap B_n) + m^*(B \cap B_n^c) = |B|.$$

So we conclude that $A_n := A \cap B_n \in \mathcal{L}$. Thus $A \in \mathcal{L}$. $\square$

<u>Rmk</u>. For general "abstract" space, there is no "box".
However, one can still develop the theory of measure. measurable sets are those that satisfy the Carathéodory criterion!

- We prove more theorems that tell us measurable sets are special.

<u>Prop</u>. Suppose $A \subset \mathbb{R}^d$ is Lebesgue measurable, $m(A) > 0$. Then $\forall\, 0 < t < 1$, $\exists$ box B s.t. $m(B \cap A) > t|B|$.

☆ We only need $m^*(A) > 0$, and the conclusion becomes $m^*(B \cap A) > t|B|$.

<u>Proof</u>. If $m(A) = +\infty$, one can introduce $A_n = A \cap B_n$. Then one can find A_n s.t. $0 < m(A_n) < +\infty$, and find the box B for A_n.

- Suppose $m(A) < \infty$. Take $0 < \varepsilon < (\frac{1}{t} - 1) m(A)$. Choose boxes B_n s.t. $A \subset \bigcup_{n=1}^{\infty} B_n$, and $\sum_n |B_n| \le m(A) + \varepsilon$.

<u>Claim</u>: $\exists k$ s.t. $t|B_k| < m(B_k \cap A)$.

[<u>Otherwise</u>: $t|B_n| \ge m(B_n \cap A)$, $\forall n \Rightarrow m(A) \le \sum m(A \cap B_n) \le t \sum |B_n|$
$\le t(m(A) + \varepsilon) < m(A)$ contradiction.]

Prop. (Steinhaus) Suppose $A \in \mathcal{L}$ and $m(A)>0$. Then the set
$$A-A := \{x-y : x, y \in A\}$$
contains a small ball $B(0,\delta)$.

Proof: Take $1-\frac{1}{2^{d+1}} < t < 1$.

By the last proposition, $\exists B$, box, s.t. $t|B| < m(B \cap A)$

Denote the shortest sidelength of B by δ.

Let $J = \{(x_1, \cdots, x_d) : |x_i| < \frac{\delta}{2}, 1 \le i \le d\}$.

We will prove $J \subset A-A$, i.e. $\forall x \in J$, the set $A \cap B$ intersects $A \cap B + \{x\}$.

[why? think about this!]

Note that by definition, the box $B+x$ contains the center of B.

It follows $m(B \cap \{B+x\}) > 2^{-d}|B|$

So $m(B \cup \{B+x\}) = m(B) + m(B+x) - m(B \cap \{B+x\})$

$$< 2|B| - 2^{-d}|B| < 2t|B|$$

But $m(A \cap B) = m(A \cap B + \{x_0\}) > t|B|$, $A \cap B \subset B \cup \{B+x\}$

$A \cap (B + \{x\}) \subset B \cup \{B+x\}$

So $A \cap B$ and $A \cap (B+x)$ must intersect. $\square$

2. Uniqueness of Lebesgue measure

Thm. Let $\tilde{m} : \mathcal{L} \to [0, +\infty]$ satisfies

(1) $\tilde{m}(\phi) = 0$

(2) $\tilde{m}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \tilde{m}(A_n)$ for disjoint $A_n \in \mathcal{L}$.

(3) $\tilde{m}(A+x) = \tilde{m}(A)$

(4) $\tilde{m}([0,1]^d) = 1$.

Then $\tilde{m} = m$.

Proof: · In lecture 2 (and PSet 1, Part 2, Problem 2) we have seen that for $E \in \mathcal{E}$, we must have $\tilde{m}(E) = m(E)$.

· By additivity, one can easily get monotonicity and subadditivity.

Now we prove the theorem step by step.

<u>Step 1</u>: $m(A)=0 \Rightarrow \tilde{m}(A)=0$.

· Let B_n be boxes s.t. $A \subset \bigcup_n B_n$ and $\sum |B_n| < \varepsilon$. Then

$$\tilde{m}(A) \le \sum_n \tilde{m}(B_n) = \sum_n |B_n| \le \varepsilon .$$

Letting $\varepsilon \to 0$, we get $\tilde{m}(A)=0$.

<u>Step 2</u>. If U is open, and $m(U)<\infty$, then $\tilde{m}(U)=m(U)$.

· Write $U=\bigcup_n B_n$ as a union of almost disjoint boxes.

Then $\mathring{B}_n$ are disjoint boxes. Let $F = U \setminus \bigcup_n \mathring{B}_n \in \mathcal{L}$.

Then $m(F) = m(U) - m(\bigcup_n \mathring{B}_n) = \sum_n m(B_n) - \sum_n m(\mathring{B}_n) = 0$.

$\Rightarrow \tilde{m}(F)=0$ by step 1.

So by additivity,

$$\tilde{m}(U) = \tilde{m}(F) + \sum_n \tilde{m}(B_n) = \sum |B_n| = m(U).$$

<u>Step 3</u>. $\tilde{m}(A) \le m(A)$.

· This is trivially true if $m(A)=\infty$.

Now suppose $m(A)<\infty$.

Choose open set $U \supset A$ s.t. $m(U)-m(A) = m(U\setminus A) < \varepsilon$.

Then $\tilde{m}(A) \le \tilde{m}(U) = m(U) \le m(A)+\varepsilon$.

Letting $\varepsilon \to 0$, we get $\tilde{m}(A) \le m(A)$.

<u>Step 4</u>: If $A \in \mathcal{L}$ is bounded, then $\tilde{m}(A)=m(A)$.

Take a box $B \supset A$. Then by additivity, $\tilde{m}(B\setminus A) = \tilde{m}(B) - \tilde{m}(A)$.

So $\tilde{m}(A) \le m(A) = m(B) - m(B\setminus A) \le \tilde{m}(B) - \tilde{m}(B\setminus A) = \tilde{m}(A)$.

<u>Step 5</u>: Let $C_n = B(0,n) \setminus B(0,n-1)$. Then

$A_n = A \cap C_n$ are bounded, and $A = \bigcup_n A_n$, disjoint union.

So $\tilde{m}(A) = \sum_n \tilde{m}(A_n) = \sum_n m(A_n) = m(A)$. $\square$