Real Analysis

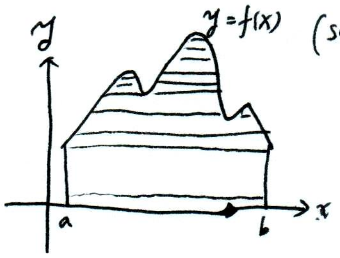
Last time.

- $A \in \mathcal{L} \iff A = G \setminus N$
  $\iff A = F \cup N$
  $\iff \forall T \subset \mathbb{R}^d, \quad m^*(T) = m^*(T \setminus A) + m^*(T \cap A)$
- "Density": $m(A) > 0 \Rightarrow \forall 0 < t < 1, \exists$ box B s.t. $m(B \cap A) > t|B|$
  [Note: can replace $m(A) > 0$ by $m^*(A) > 0$.]
  $\longrightarrow m(A) > 0 \Rightarrow A - A$ contains a small ball $B(0, \delta)$.
- Uniqueness: m is the only translation-invariant, normalized measure on $\mathcal{L}$.

Today: Measurable functions

1. Measurable functions

- Recall (Lecture 1): Lebesgue's idea to define integral.



WANT: $\int_a^b f(x)\,dx = \int_0^\infty m(\{x : f(x) > t\})\,dt$.

Def. We say a $\underbrace{\text{function}}_{}$ (d-variable) f is <u>measurable</u> if for any $t \in \mathbb{R}$, the set
$$\{x : f(x) > t\}$$
is a measurable set in $\mathbb{R}^d$.

Remark. ① We will allow the domain of f to be a subset $A \subset \mathbb{R}^d$, in which case we require A to be a measurable set, and we say f is a measurable function on A.

② We allow the value of f to be $\pm\infty$ at some points. ($\mathbb{R}^*$-valued)
We say f is finite-valued if $-\infty < f(x) < \infty$ for all x (or $\mathbb{R}$-valued)
[However, it is still possible that f is unbounded, e.g. $f(x) = x^2$ on $\mathbb{R}$.]

③ We say two functions f and g are equal almost everywhere, and denote
$$f = g \quad \text{a.e.}$$
if the set $\{x \mid f(x) \neq g(x)\}$ is a null set (i.e. measure 0 set).
e.g. $f(x) \equiv 0$, $g(x) = \delta_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$, then $f = g$ a.e.
[Similarly we say a proposition holds almost everywhere, if it only fails on a null set.]

Example. $A \subset \mathbb{R}^d$ is measurable $\iff \chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ is Lebesgue measurable.

- There are many equivalent definitions. For example,

Prop. Let $f: A \subset \mathbb{R}^d \to \mathbb{R}$ be a function defined on a measurable set A. T.F.A.E.

(1) $\forall t \in \mathbb{R}$, the set $\{x : f(x) > t\}$ is Lebesgue measurable.

(2) $\forall t \in \mathbb{R}$, ---------- $\geq$ ----------

(3) ---------- $<$ ----------

(4) ---------- $\leq$ ----------

Cor. If f is measurable, then for any t, $f^{-1}(t) \in \mathcal{L}$.

Proof. (1)$\Leftrightarrow$(4): A set is measurable if and only if its complement is measurable.

(2)$\Leftrightarrow$(3): SAME reason.

~~(1)$\Rightarrow$(2)~~:

(2)$\Rightarrow$(1): $\{x : f(x) > t\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq t + \frac{1}{n}\}$.

(4)$\Rightarrow$(3): $\{x : f(x) < t\} = \bigcup_{n=1}^{\infty} \{x : f(x) \leq t - \frac{1}{n}\}$. $\square$

"New" trick: Write one set as the union of countably many sets.

Note (a) We are NOT saying that for a single t_0, $\{x : f(x) > t_0\} \in \mathcal{L} \Leftrightarrow \{x : f(x) \geq t_0\} \in \mathcal{L}$.

(b) In today's problem set, you will add two more equivalent definition to the list in the above proposition:

(5) $\forall$ open set $U \subset \mathbb{R}$, the set $f^{-1}(U)$ is Lebesgue measurable

(6) $\forall$ closed set $F \subset \mathbb{R}$, the set $f^{-1}(F)$ is ----------.

Recall: A function f is continuous if and only if $\forall$ open set U, $f^{-1}(U)$ is open

[This will be the definition of continuity for functions defined on some abstract space.]

As a consequence, we immediatly get

Cor. (1) Any continuous function is measurable

(2) If $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous, $f : A \subset \mathbb{R}^d \to \mathbb{R}$ is measurable, then $\Phi \circ f : A \subset \mathbb{R}^d \to \mathbb{R}$ is measurable.

Proof: (1) Obvious.

(2) Let $U = \Phi^{-1}((t, +\infty))$. Then U is open since Φ is continuous.

Thus $(\Phi \circ f)^{-1}((t, +\infty)) = f^{-1} \circ \Phi^{-1}((t, +\infty)) = f^{-1}(U)$ is measurable. $\square$

~~Rmk~~ WARNING: One can construct continuous function Φ and continuous function f such that $f \circ \Phi$ is NOT measurable.

In particular, it is possible that the composition of measurable functions is no longer measurable.

2. Operations on measurable functions

- First we show that the set of measurable ($\mathbb{R}$-valued) functions form an algebra.

Prop: If f and g are $\mathbb{R}$-valued measurable functions defined on $A \subset \mathbb{R}^d$, then cf, $f \pm g$, $f \cdot g$ are all $\mathbb{R}$-valued measurable functions on A.

Proof. (1) If $c > 0$, then $\{x : cf(x) > t\} = \{x : f(x) > \frac{t}{c}\} \Rightarrow cf$ measurable

If $c < 0$, then $\{x : cf(x) > t\} = \{x : f(x) < \frac{t}{c}\} \Rightarrow$ ....

If $c = 0$, obviously $cf = 0$ is measurable.

(2) [This is NOT obvious!] [Idea: Write one set as the union of countably many!]

Suppose $f(x) + g(x) > t$. Then $f(x) > t - g(x)$.

One can pick a rational number $r \in \mathbb{Q}$ s.t. $f(x) > r > t - g(x)$

i.e. $f(x) > r$ AND $g(x) > t - r$.

This observation gives us a decomposition

$$\{x : f(x) + g(x) > r\} = \bigcup_{r \in \mathbb{Q}} \left(\{x : f(x) > r\} \cap \{x : g(x) > t - r\} \right)$$

[We indeed showed $\subset$, but $\supset$ is obvious.]

Now from definition we see $f + g$ is measurable.

(3) To show fg is measurable, we first notice that

f is measurable $\Rightarrow f^2$ is measurable,

Since (non-negative)

$$\{x : f^2(x) > r\} = \{x : f(x) > \sqrt{r}\} \cup \{x : f(x) < -\sqrt{r}\}.$$

Now fg is measurable, since

$$fg = \left[(f+g)^2 - f^2 - g^2\right] / 2,$$

each item is measurable. $\square$

Rmk: Usually we don't add $\overline{\mathbb{R}}$-valued functions, since $(-\infty) + (+\infty)$ makes no sense.

• Second, we look at the limits of measurable functions.

<u>Thm</u>: Suppose f_n are measurable, and $f_n(x) \to f(x)$, $\forall x$. Then f is measurable.

[<u>Recall</u>: continuous functions may converge to a discontinuous function.
So measurable functions behave much nicer than continuous functions under the limit operation.]

<u>Proof</u>: [Idea: "decompose"]

Suppose $\underset{\lim_{n\to\infty} f_n(x)}{\underset{\parallel}{f(x)}} > r$. Then $\exists k$ and N s.t. $\forall n > N$, $f_n(x) > r + \frac{1}{k}$.

[Trick: $\exists \leftrightarrow \cup$, $\forall \leftrightarrow \cap$]

So we get

$$\{x: f(x) > r\} = \bigcup_k \bigcup_N \bigcap_{n\ge N} \{x: f_n(x) > r + \tfrac{1}{k}\}.$$

[Again, we showed "$\subset$", while "$\supset$" is obvious.

Since a countable union/intersection of Lebesgue measurable sets remains Lebesgue measurable, we get the conclusion. $\square$

In fact, we can say more, we don't really need f_k to be convergent!

<u>Prop</u>: Let f_n be a sequence of measurable functions. Then the following functions are all measurable: $\sup_n f_n(x)$, $\inf_n f_n(x)$, $\limsup_{n\to\infty} f_n(x)$, $\liminf_{n\to\infty} f_n(x)$.

<u>Proof</u>:
- $\sup_n f_n(x) > t \iff \exists n$ s.t. $f_n(x) > t$

  So $\{x: \sup_n f_n(x) > t\} = \bigcup_n \{x: f_n(x) > t\}$
- $\inf_n f_n(x)$ is similar.
- $\limsup_{n\to\infty} f_n(x) > t \iff \forall N, \exists n \geq N$ s.t. $f_n(x) > t$.

  So
  $$\{x: \limsup_{n\to\infty} f_n(x) > t\} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x: f_n(x) > t\}.$$
- $\liminf_{n\to\infty} f_n(x)$ is similar. $\square$

<u>Cor</u>: If f is measurable, so is $|f|$.