Real Analysis

Lecture 7
03/19/2018

Last time

- A function f is measurable $\Leftrightarrow$ $\forall t \in \mathbb{R}$, the set $\{x : f(x) > t\} \in \mathcal{L}$
  - $\Leftrightarrow$ ---------- $\geq$ ----
  - $\Leftrightarrow$ ---------- $<$ ----
  - $\Leftrightarrow$ ---------- $\leq$ ----
  - $\Leftrightarrow$ $\forall$ open set $U \subset \mathbb{R}$, the set $f^{-1}(U) \in \mathcal{L}$
  - $\Leftrightarrow$ $\forall$ closed set $F \subset \mathbb{R}$, the set $f^{-1}(F) \in \mathcal{L}$
  - $\Leftrightarrow$ $\forall$ Borel set $A \subset \mathbb{R}$, the set $f^{-1}(A) \in \mathcal{L}$.

decompose one set to a countable union of sets

Trick: $\forall \rightsquigarrow \bigcap$, $\exists \rightsquigarrow \bigcup$

- Examples of measurable function.
  - χ_A ($A \in \mathcal{L}$) · monotone functions · continuous functions
  - $\Phi \circ f$ (where Φ is continuous, f is measurable) ($\rightsquigarrow |f|$, f^2 etc.)
  - cf, $f+g$, $f \cdot g$, (f, g are measurable)
  - $\lim\limits_{n\to\infty} f_n$, $\limsup\limits_{n\to\infty} f_n$, $\liminf\limits_{n\to\infty} f_n$, $\sup\limits_n f_n$, $\inf\limits_n f_n$ (f_n: a sequence of measurable functions)
- The phrase "almost everywhere" (a.e.) $\Leftrightarrow$ "holds except for a null set". e.g.
  - $f = g$ a.e. $\Leftrightarrow$ $m(\{x : f(x) \neq g(x)\}) = 0$
  - f is finite a.e. $\Leftrightarrow$ $m(\{x : f(x) = \pm\infty\}) = 0$.
  - $f_n \to f$ a.e. $\Leftrightarrow$ $m(\{x : \lim\limits_{n\to\infty} f_n(x) \neq f(x)\}) = 0$.

Today: Various modes of convergence

1. Convergence in measure

- Let's first take a closer look at a.e. convergence.

($\lim\limits_{n\to\infty} f_n(x) = f(x)$: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall k > N$, $|f_k(x) - f(x)| < \varepsilon$.)

$\longleftrightarrow$ $\lim\limits_{n\to\infty} f_n(x) \neq f(x)$: $\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists k > N$ s.t. $|f_k(x) - f(x)| > \varepsilon$.

$\Leftrightarrow$ $\exists \varepsilon > 0$ s.t. $x \in \bigcap\limits_{N=1}^{\infty} \bigcup\limits_{k=N}^{\infty} A_k(\varepsilon)$, where

$$\boxed{A_k(\varepsilon) = \{x : |f_k(x) - f(x)| > \varepsilon\}} \quad (\star)$$

- **Def.** For any sequence A_k of sets, the upper limit is

$$\limsup_{k\to\infty} A_k := \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} A_k.$$

[Similarly one can define

$$\liminf_{k\to\infty} A_k := \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A_k.$$]

[Compare. For a sequence of real numbers x_n:

$$\limsup_{k\to\infty} x_k = \inf_{n\geq 1} \sup_{k\geq n} x_k$$

$$\liminf_{k\to\infty} x_k = \sup_{n\geq 1} \inf_{k\geq n} x_k.$$]

Prop.: $\limsup_{k\to\infty} A_k = \{x : x\in A_k \text{ for infinitely many } A_k\}$.

Proof. $x\in \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} A_k \Leftrightarrow \forall N,\ x\in \bigcup_{k=N}^{\infty} A_k$

$\Leftrightarrow \forall N,\ \exists k>N$ s.t. $x\in A_k$

$\Leftrightarrow x\in A_k$ for infinitely many A_k. $\square$

The following lemma is useful:

Borel-Cantelli lemma. Suppose $A_k\in\mathcal{L}$, and $\sum_{k=1}^{\infty} m(A_k)<\infty$.
Then $A_\infty := \limsup_{k\to\infty} A_k \in\mathcal{L}$, and $m(A_\infty)=0$.

[Compare: $\sum_{n=1}^{\infty} a_n<\infty \Rightarrow \lim_{n\to\infty} a_n=0$.] Proof: Exercise. $\square$

- Now let's turn back to a.e. convergence. Using the language of limsup, we have

Prop.: $f_n\to f$ a.e. $\Leftrightarrow \forall\varepsilon>0,\ m(\limsup_{k\to\infty} A_k(\varepsilon))=0$.

Proof. - We have just seen that

$f_n\to f$ a.e. $\Leftrightarrow m(\bigcup_{\varepsilon>0}\limsup_{k\to\infty} A_k(\varepsilon))=0$.

So $\forall\varepsilon>0$, by monotonicity,

$m(\limsup_{k\to\infty} A_k(\varepsilon))=0$.

- Conversely, suppose $m(\limsup_{k\to\infty} A_k(\varepsilon))=0$ for $\forall\varepsilon>0$. We can pick a sequence $\varepsilon_j\to 0$. Observe that $\varepsilon_1<\varepsilon_2 \Rightarrow A_k(\varepsilon_1)\supset A_k(\varepsilon_2)$

$\Rightarrow \limsup_{k\to\infty} A_k(\varepsilon_1)\supset \limsup_{k\to\infty} A_k(\varepsilon_2)$.

$\Rightarrow \bigcup_{\varepsilon>0}\limsup_{k\to\infty} A_k(\varepsilon) = \bigcup_{j=1}^{\infty}\limsup_{k\to\infty} A_k(\varepsilon_j)$.

So by sub-additivity,

$m(\bigcup_{\varepsilon>0}\limsup_{k\to\infty} A_k(\varepsilon)) = m(\bigcup_{j=1}^{\infty}\limsup_{k\to\infty} A_k(\varepsilon_j)) \le \sum_{j=1}^{\infty} m(\limsup_{k\to\infty} A_k(\varepsilon_j))=0$.

It follows that $f_n\to f$ a.e. $\square$

- Motivated by this description of a.e. convergence, we define

Def.: Let f_n be a sequence of a.e. finite and measurable functions. [We will always assume this in todays lecture]
We say $f_n\to f$ in measure, if $\forall\varepsilon>0$,

$\lim_{k\to\infty} m(A_k(\varepsilon))=0$.

Rmk: In the context of probability theory, "a.e. convergence" is often referred to as "almost sure convergence", and "convergence in measure" is often referred to as "convergence in probability".

- By definition one might have the impression that a.e. convergence is stronger than convergence in measure. This is almost true:

Prop. Suppose $m(A)<\infty$. Then $f_n \to f$ a.e. $\Rightarrow$ $f_n \to f$ in measure.

Proof. Fix any $\varepsilon>0$.

Consider the monotone decreasing sequence of sets

$$(A\supset)\ \bigcup_{k=1}^{\infty} A_k(\varepsilon) \supset \bigcup_{k=2}^{\infty} A_k(\varepsilon) \supset \bigcup_{k=3}^{\infty} A_k(\varepsilon) \supset \cdots \supset \bigcup_{k=N}^{\infty} A_k(\varepsilon) \supset \cdots$$

Since $m(A)<\infty$, we can apply the (downward) monotone convergence theorem (PSet 2, Part 2, problem 4) to conclude

$$\lim_{N\to\infty} m\Big(\bigcup_{k=N}^{\infty} A_k(\varepsilon)\Big) = m\Big(\bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} A_k(\varepsilon)\Big) = m\big(\limsup_{k\to\infty} A_k(\varepsilon)\big) = 0.$$

since $f_n \to f$ a.e.

Since $A_N(\varepsilon) \subset \bigcup_{k=N}^{\infty} A_k(\varepsilon)$, we get (by monotonicity)

$$\lim_{N\to\infty} m(A_N(\varepsilon)) = 0.$$

i.e. $f_n \to f$ in measure. $\square$

- **Rmk**: One can't drop the condition $m(A)<\infty$ in the proposition.

Example: ~~For example~~ Take $f_n(x)=\frac{x}{n}$, $x\in\mathbb{R}$.

Then $\forall x\in\mathbb{R}$ fixed, we have $f_n(x)\to 0$ as $n\to\infty$.

So we have pointwise (and thus a.e.) convergence $f_n\to 0$.

Hower, for any $\varepsilon>0$,

$$m(\{x : |f_n(x)|>\varepsilon\}) = m(\{x : |x|>n\varepsilon\}) = \infty.$$

So $f_n \not\to 0$ in measure.

Example: For $n=2^\ell+k<2^{\ell+1}$, we let $E_n=\left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right]$.

Let $f_n(x)=\chi_{E_n}(x)$.

[e.g. $f_1(x)=\chi_{[0,1]}$, $f_2(x)=\chi_{[0,\frac12]}$, $f_3(x)=\chi_{[\frac12,1]}$, $f_4(x)=\chi_{[0,\frac14]}$, $f_5(x)=\chi_{[\frac14,\frac24]}$, $\cdots$, $f_{17}(x)=\chi_{[\frac{1}{16},\frac{2}{16}]}$, $\cdots$]

Then $f_n\to 0$ in measure (since $m(A_n(\varepsilon))=\frac{1}{2^\ell}\to 0$ as $n\to\infty$).

However, $f_n \not\to 0$ a.e. $\quad A_n(\varepsilon)=\left[\frac{k}{2^\ell},\frac{k+1}{2^\ell}\right]$

[Since $\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n(\varepsilon) = \bigcap_{N=1}^{\infty}[0,1]=[0,1]$]

(In fact: $\forall x\in[0,1]$, $f_n(x)\not\to 0$.)

- However, if we take the subsequence $f_1, f_2, f_4, f_8, f_{16}, \cdots$, then we can easily see $f_{2^k}\to 0$ a.e.

It turns out that this "convergent subsequence" phenomina holds in general.

• **Thm** (Riesz) Suppose $f_n \to f$ in measure. Then there is a subsequence f_{n_k} of f_n s.t. $f_{n_k} \to f$ a.e.

Proof: $\forall k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ s.t.

$$m\left(A_n\left(\tfrac{1}{2^k}\right)\right) < \frac{1}{2^k}, \quad \forall n \geq n_k.$$

Note that we can choose these n_k's s.t. $n_1 < n_2 < n_3 < \cdots$.

By our choice, we have

$$\sum_{k=1}^{\infty} m\left(A_{n_k}\left(\tfrac{1}{2^k}\right)\right) < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

So according to Borel-Cantelli Lemma,

$$m\left(\limsup_{k\to\infty} A_{n_k}\left(\tfrac{1}{2^k}\right)\right) = 0.$$

Now ~~for any~~ $x \notin \limsup_{k\to\infty} A_{n_k}\left(\frac{1}{2^k}\right)$, $\Leftrightarrow$ x sits in finitely many $A_{n_k}\left(\frac{1}{2^k}\right)$'s

$\Leftrightarrow \exists K$ s.t. $\forall k > K$, $x \notin A_{n_k}\left(\frac{1}{2^k}\right)$.

$\Leftrightarrow \exists K$ s.t. $\forall k > K$, $|f_{n_k}(x) - f(x)| < \frac{1}{2^k}$.

$\Leftrightarrow f_{n_k}(x) \to f(x)$

So $f_{n_k} \to f$ on $A \setminus \limsup_{k\to\infty} A_{n_k}\left(\frac{1}{2^k}\right)$.

In other words, $f_{n_k} \to f$ a.e. $\square$.

Cor: If $f_n \to f$ in measure, then f is a measurable function.

Proof. If $f_n \to f$ in measure, then $\exists$ subsequence $f_{n_k} \to f$ a.e.

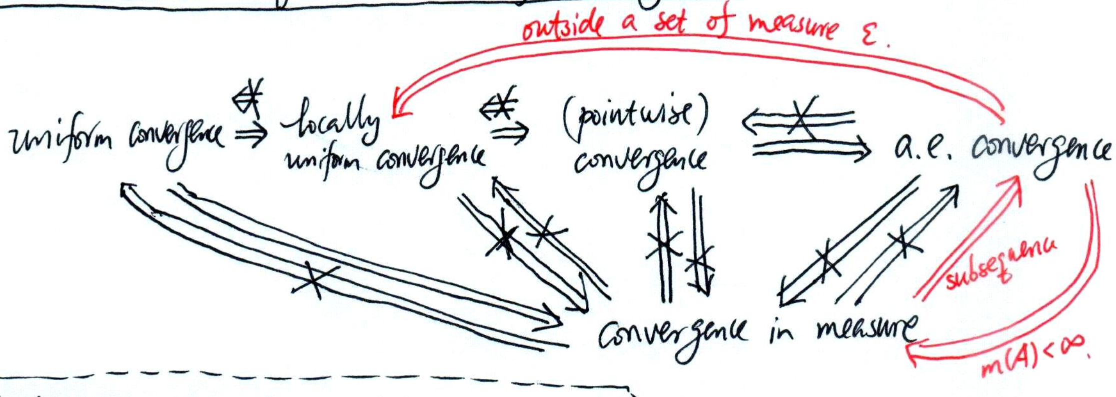
Note: So we proved that if $f_n \to f$ a.e., then f is measurable.

So $\exists$ null set $A_0 \subset A$ s.t. $f_{n_k} \to f$ on $A \setminus A_0$

So f is measurable on $A \setminus A_0$.

Since $m(A_0) = 0$, we see f is measurable on A. (why?) $\square$.

Relations between different modes of convergence.



black arrows ⇒ and ⇏: definition or examples.

red arrows ⇒, three theorems.

2. Uniform convergence

- Recall: • $f_n \to f$ uniformly on $A \Leftrightarrow \forall \varepsilon>0, \exists N$ s.t. $|f_n(x)-f(x)|<\varepsilon$ for $\forall n\geq N$ and $\forall x\in A$.

  • uniform convergence mode is one of the best convergence mode.
  However, if a set is unbounded, then "uniform convergence" is too strong.

Def: We say $f_n \to f$ locally uniformly if for any bounded set E, $f_n \to f$ uniformly on E.

Rmk: One can show: $f_n \to f$ locally uniformly $\Leftrightarrow \forall x, \exists$ a nghd U of x s.t. $f_n \to f$ uniformly on U.
(Usually when we say a proposition holds locally, if $\forall x, \exists$ a nghd U of x s.t. the proposition holds on U.)

- Example. Let $f_n(x)=\frac{x}{n}$, $x\in\mathbb{R}$.
  Then $f_n \to 0$ locally uniformly (and thus $f_n\to 0$, $f_n \to 0$ a.e.)
  But $f_n \not\to 0$ uniformly, $f_n \not\to 0$ in measure.

Example: Let $f_n(x)=\frac{1}{nx}\chi_{\mathbb{R}^+} = \begin{cases} \frac{1}{nx}, & x>0 \\ 0, & x\leq 0. \end{cases}$

Then $f_n \to 0$ pointwise, (and thus $f_n \to 0$ a.e.)
and $f_n \to 0$ in measure.
But $f_n \not\to 0$ locally uniformly (and thus $f_n \not\to 0$ uniformly).

- Thm (Egorov) Suppose $f_n \to f$ a.e. Then $\forall\varepsilon>0$, $\exists$ a measurable set A_ε with $m(A_\varepsilon)<\varepsilon$, s.t. $f_n \to f$ locally uniformly on $A\setminus A_\varepsilon$.

Proof: First take A_0 with $m(A_0)=0$ s.t. $f_n\to f$ on $A\setminus A_0$.
Denote
$$A_{N,m}=\bigcup_{n=N}^{\infty}\left(A_n(\tfrac{1}{m})\cap(A\setminus A_0)\right)$$
$$=\{x\in A\setminus A_0 : |f_n(x)-f(x)|\geq \tfrac{1}{m} \text{ for some } n\geq N\}$$
Then $x\in \bigcap_{N=1}^{\infty} A_{N,m} \Leftrightarrow \forall N, \exists n\geq N$ s.t. $|f_n(x)-f(x)|\geq\frac{1}{m}$
$\Rightarrow f_n(x)\not\to f$.

It follows that $\bigcap_{N=1}^{\infty} A_{N,m}=\phi$.
On the other hand, we have a decreasing sequence of sets
$$A_{1,m}\supset A_{2,m}\supset A_{3,m}\supset\cdots\supset A_{N,m}\supset\cdots$$
$\rightsquigarrow A_{1,m}\cap B(0,m)\supset A_{2,m}\cap B(0,m)\supset A_{3,m}\cap B(0,m)\supset\cdots\supset A_{N,m}\cap B(0,m)\supset\cdots$

This is a decreasing sequence of sets of finite measure.

So by downward monotone convergence thm (PSet 2, Part 2, Problem 4)

$$\lim_{N\to\infty} m(A_{N,m} \cap B(0,m)) = m\left(\bigcap_{N=1}^{\infty}(A_{N,m} \cap B(0,m))\right) = m(\emptyset) = 0.$$

In particular, one can find $N_m > 0$ s.t.

$$m(A_{N,m} \cap B(0,m)) < \frac{\varepsilon}{2^m}, \quad \forall N \geq N_m.$$

Now let

$$A_\varepsilon = \bigcup_{m=1}^{\infty} (A_{N_m,m} \cap B(0,m)) \cup A_0 \quad \in \mathcal{L}.$$

Then $m(A_\varepsilon) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} + 0 = \varepsilon.$

Moreover, by construction we have

$$|f_n(x) - f(x)| < \frac{1}{m},$$

holds for $\forall m \geq 1$, $\forall x \in (A \setminus A_\varepsilon) \cap B(0,m)$ and $\forall n \geq N_m$.

So for any bounded set $E \subset A \setminus A_\varepsilon$, we have

$$f_n \to f \text{ uniformly in } E.$$

[To see this, for $\forall \varepsilon > 0$, one just take m large enough s.t. $\frac{1}{m} < \varepsilon$ and $E \subset B(0,m)$.] $\square$

Rmk: Egorov's thm explains Littlewood's 3rd principle:

Every convergent sequence of measurable functions is nearly uniformly convergent.

Cor. (Egorov) (different version) Suppose $m(A) < \infty$.
If $f_n \to f$ a.e., then $\forall \varepsilon > 0$, $\exists A_\varepsilon \in \mathcal{L}$ with $m(A_\varepsilon) < \varepsilon$
s.t. $f_n \to f$ uniformly on $A \setminus A_\varepsilon$.

Proof: By the Egorov thm we just proved, one can find $\tilde{A}_\varepsilon$ with $m(\tilde{A}_\varepsilon) < \frac{\varepsilon}{2}$,

s.t. $f_n \to f$ locally uniformly on $A \setminus \tilde{A}_\varepsilon$.

Since $m(A) < \infty$, and

$$A \cap B(0,1) \subset A \cap B(0,2) \subset \cdots \subset A \cap B(0,N) \subset \cdots$$

by monotone convergence,

$$\lim_{N\to\infty} m(A \cap B(0,N)) = m\left(\bigcup_N (A \cap B(0,N))\right) = m(A) < \infty.$$

So $\exists N$ s.t. $m(A \cap B(0,N)) \geq m(A) - \frac{\varepsilon}{2}$.

It follows that $f_n \to f$ uniformly on $\underbrace{(A \setminus \tilde{A}_\varepsilon) \cap B(0,N)}_{= A \setminus A_\varepsilon,\ A_\varepsilon = \tilde{A}_\varepsilon \cup (A \cap B(0,N)^c)}$, and $m(A_\varepsilon) < \varepsilon$. $\square$

3. Approximation by simple functions

- **Def.** A *simple function* is a function of the form
$$f(x) = c_1 \chi_{A_1}(x) + \cdots + c_n \chi_{A_n}(x),$$
where $c_1, \cdots, c_n$ are numbers, and $A_1, \cdots, A_n \in \mathcal{L}$ are disjoint.

Rmk. *Step function* are special simple functions where A_i's are boxes.

e.g. $\chi_{\mathbb{Q}}$ is a simple function, but not a step function.

- **Thm.** A function f defined on a measurable set A is a measurable function if and only if there exits a sequence f_n of simple functions s.t.
$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in A.$$
Moreover, (a) If $f \ge 0$ on A, then one can take $0 \le f_n \le f_{n+1} \le \cdots$

(b) In general, one can take $|f_n| \le |f|$.

(c) If $f(x)$ is bounded, then $f_n \to f$ uniformly.

Proof. (a) First assume $f \ge 0$.

Fix $n \in \mathbb{N}$. Define
$$f_n(x) = \begin{cases} \frac{k-1}{2^n}, & \text{if } \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n} \quad (k = 1, 2, \cdots, n \cdot 2^n) \\ n, & \text{if } f(x) \ge n. \end{cases}$$

Since f is measurable, each f_n is a simple function, and $f_n(x) \le f(x)$, $\forall x$.

It is also clear that
$$f_n(x) \le f_{n+1}(x) \quad \text{and} \quad \begin{cases} f_{n+1}(x) - f_n(x) \le \frac{1}{2^{n+1}} & \text{if } f(x) \le n \\ f_{n+1}(x) - f_n(x) \le 1 & \text{if } f(x) > n. \end{cases}$$

It follows that

(i) $f_n(x) \to f(x)$

(ii) If f is bounded, i.e. $f(x) \le M$ for some M, then
$$f_{n+1}(x) - f_n(x) \le \frac{1}{2^n}$$
for $n > M$, and thus the convergence is uniform.

(b) For general case, let $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$.

Then both f^+ and f^- are measurable and non-negative.

So by part (a), $\exists$ $\underbrace{g_n(x)}_{\text{simple function}} \le g_{n+1}(x) \le \cdots$ and $h_n(x) \le h_{n+1}(x) \le \cdots$

s.t. $\lim_{n\to\infty} g_n(x) = f^+(x)$, $\lim_{n\to\infty} h_n(x) = f^-(x)$.

s.t. $g_n = 0$ if $f^+ = 0$
$h_n = 0$ if $f^- = 0$

Let $f_n = g_n - h_n$. Then $f_n \to f^+ - f^- = f$.

Observe: (i) If $f^- = 0$, then $f_n = g_n \le f^+$; If $f^+ = 0$, then $f_n = h_n \le f^-$; $\Rightarrow |f_n| \le |f|$.

(ii) If f^+, f^- are bounded, then $g_n \to f^+$, $h_n \to f^-$ uniformly $\Rightarrow f_n \to f$ uniformly. $\square$