Real Analysis

Lecture 8
03/22/2018

习题课：周日答疑时间
every 2 weeks

Last time

- f_n = a sequence of a.e. finite functions, measurable $\rightsquigarrow f(x)$.

  $A_k(\varepsilon) = \{x : |f_k(x) - f(x)| > \varepsilon\}$.

  $\limsup\limits_{k\to\infty} A_k = \bigcap\limits_{N=1}^{\infty} \bigcup\limits_{k=N}^{\infty} A_k$

  Different modes of convergence.

  (1) (pointwise) convergence: $f_n \to f \iff \lim\limits_{n\to\infty} f_n(x) = f(x),\ \forall x$

  $\iff \limsup\limits_{k\to\infty} A_k(\varepsilon) = \phi,\ \forall \varepsilon > 0$

  (2) a.e. convergence: $f_n \to f$ a.e. $\iff m(\{x : \lim\limits_{n\to\infty} f_n(x) \neq f(x)\}) = 0$.

  $\iff m(\limsup\limits_{k\to\infty} A_k(\varepsilon)) = 0,\ \forall \varepsilon > 0$.

  (3) convergence in measure: $f_n \to f$ in measure $\iff \lim\limits_{k\to\infty} m(A_k(\varepsilon)) = 0,\ \forall \varepsilon > 0$.

  (4) uniform convergence: $f_n \to f$ uniformly $\iff \forall \varepsilon > 0, \exists N$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for $\forall n \geq N$ and $\forall x$.

  $\iff \forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N$, $A_k(\varepsilon) = \phi$.

  (5) locally uniform convergence: $f_n \to f$ locally uniformly $\iff \forall$ bounded set E, $f_n \to f$ uniformly on E.

  $\iff \forall$ bound set E, $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N$, $A_k(\varepsilon) \cap E = \phi$.

$$\text{u.c.} \Rightarrow \text{l.u.c.} \Rightarrow \text{p.c.} \Rightarrow \text{a.e.c.}$$

$$\text{a.e.c.} \xrightarrow{\text{Egorov: outside } \varepsilon\text{-set}} \text{l.u.c.}$$

$$\text{u.c.} \Rightarrow \text{c. in m.}$$

$$\text{a.e.c.} \xrightarrow{m(A) < \infty} \text{c. in m.}$$

$$\text{c. in m.} \xrightarrow{\text{Riesz: subseq.}} \text{a.e.c.}$$

Note. In all these cases, the limit f is measurable.

"almost uniform convergence"
=
uniform convergence outside a set of measure ε.

- ~~Simple function.~~ $\cancel{f = c_1\chi_{A_1} + c_2\chi_{A_2} + \cdots + c_N\chi_{A_N}}$

  - ~~f is measurable $\iff \exists$ simple functions f_n s.t. $f_n \to f$ (pointwise)~~ ~~with $0 \leq |f_1| \leq |f_2| \leq \cdots \leq |f|$~~
    - ~~If $f \geq 0$, then can take $0 \leq f_1 \leq f_2 \leq f_3 \leq \cdots \leq f$~~
    - ~~If f is bounded, then $f_n \to f$ uniformly~~

Today. Approximations of measurable functions

1. Approximation by simple functions

- **Def.** A simple function is a function of the form

  $$f(x) = c_1\chi_{A_1}(x) + c_2\chi_{A_2}(x) + \cdots + c_n\chi_{A_n}(x),$$

  where $c_1, \cdots, c_n$ are numbers, and $A_1, \cdots, A_n \in \mathcal{L}$ are distinct measurable sets.

  Example. $\cdot\ \chi_{\mathbb{Q}}$.

  - Step function = simple function on $\mathbb{R}$, with A_i = intervals.

    (or in general, A_i = boxes)

• **Thm 1** If $f \geq 0$ is a non-negative measurable function defined on $A \in \mathcal{L}$.
Then there exists an increasing sequence of non-negative, ~~increasing~~ simple functions
$$0 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots \leq f$$
such that $\lim_{n\to\infty} f_n(x) = f(x), \ \forall x \in A$.
Moreover, if f is bounded, then the convergence is uniform.

Proof: Fix $n \in \mathbb{N}$. Define
$$f_n(x) = \begin{cases} \frac{k-1}{2^n}, & \text{if } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \quad (k=1,2,\cdots,n 2^n) \\ n, & \text{if } f(x) \geq n. \end{cases}$$
Since f is measurable, each set $\{x : \frac{k-1}{2^n} \leq f < \frac{k}{2^n}\}$ and $\{x : f(x) \geq n\}$ are measurable.
So each f_n is a simple function. By definition it is clear that
$$0 \leq f_n(x) \leq f(x), \qquad \forall x.$$
Moreover, we have

(1) If $\underline{f(x) < n}$, then $\exists k \leq n \cdot 2^n$ s.t. $\frac{2k-2}{2^{n+1}} = \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} = \frac{2k}{2^{n+1}}$
$$\Rightarrow f_n(x) = \frac{k-1}{2^n}, \quad f_{n+1}(x) = \frac{2k-2}{2^{n+1}} = \frac{k-1}{2^n} \text{ or } f_{n+1}(x) = \frac{2k-1}{2^{n+1}}.$$
$$\Rightarrow 0 \leq f_{n+1}(x) - f_n(x) \leq \frac{1}{2^{n+1}}$$

(2) If $f(x) \geq n$, then $f_n(x) = n$, while $n = \frac{n \cdot 2^{n+1}}{2^{n+1}} \leq f(x)$ implies
$$f_{n+1}(x) = n \text{ or } n + \frac{1}{2^{n+1}} \text{ or } n + \frac{2}{2^{n+1}} \text{ or } \cdots \text{ or } n+1.$$
$$\Rightarrow 0 \leq f_{n+1}(x) - f_n(x) \leq 1.$$

So we get
$$0 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots \leq f$$
and $\lim_{n\to\infty} f_n(x) = f(x), \ \forall x \in A$.

to see this, discuss 2 cases: $f(x) < \infty$, $f(x) = \infty$

Moreover, if $|f(x)| < M$, $\forall x$, then only case (1) can happen for $n > M$.
So the convergence is uniform. $\square$

Thm 2: For any measurable function f defined on $A \in \mathcal{L}$,
there exists a sequence of simple functions with
$$0 \leq |f_1| \leq |f_2| \leq \cdots \leq |f_n| \leq \cdots \leq |f|$$
such that $\lim_{n\to\infty} f_n(x) = f(x), \ \forall x \in A$.
Moreover, if f is bounded, then the convergence is uniform.

Proof: Let $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$.
Then both f^+ and f^- are measurable, non-negative, and $f = f^+ - f^-$.
By Thm 1, there exists simple functions
$$0 \leq g_1(x) \leq g_2(x) \leq \cdots \leq f^+(x) \quad \text{and} \quad 0 \leq h_1(x) \leq h_2(x) \leq \cdots \leq f^-(x)$$
such that $\lim_{n\to\infty} g_n(x) = f^+(x)$, $\lim_{n\to\infty} h_n(x) = f^-(x)$.

Let $f_n(x) = g_n(x) - h_n(x)$. Then $\lim_{n\to\infty} f_n(x) = f^+(x) - f^-(x) = f(x)$.

Note: • if $f(x) > 0$, then $f^-(x) = 0 \Rightarrow h_n(x) = 0$, $h_{n+1}(x) = 0$ $\Rightarrow \underset{|f_n(x)|}{f_n(x)} = g_n(x) \le g_{n+1}(x) = \underset{|f_{n+1}(x)|}{f_{n+1}(x)} \le f^+(x) = |f(x)|$

• if $f(x) < 0$, then $f^+(x) = 0 \Rightarrow g_n(x) = 0$, $g_{n+1}(x) = 0$ $\Rightarrow |f_n(x)| = h_n(x) \le h_{n+1}(x) = |f_{n+1}(x)| \le f^-(x) = |f(x)|$

So we always has

$$0 \le |f_1| \le |f_2| \le \cdots \le |f_n| \le \cdots \le |f|$$

Moreover, if f is bounded, then both f^+ and f^- are bounded.

So $g_n \to f^+$ uniformly, $h_n \to f^-$ uniformly.

It follows that $f_n \to f$ uniformly. □

Cor: f is measurable $\Leftrightarrow$ f is the limit of a sequence of simple functions.

Cor. If f is measurable, then $\exists$ a sequence of step functions $f_1, f_2, \cdots$ s.t. $f_n \to f$ a.e.

Proof: Exercise. (Hint: use PSet 2, Part 2, Problem 2 "nearly open") □

2. Approximation by continuous functions

• In PSet 4, Part 1, Problem 4: one can "approximate" simple functions on $\mathbb{R}$ via continuous function. As a consequence, one can "approximate" any measurable function via continuous function.

The exact statement (in any dimension) is

Thm. (Lusin) Let f be any a.e. finite and measurable function defined on $A \in \mathcal{L}$. Then $\forall \varepsilon > 0$, $\exists$ closed set $F \subset A$ with $m(A \setminus F) < \varepsilon$, such that f is continuous on F.

Rmk. This explains Littlewood's 2nd principle:

"Every measurable function is nearly continuous".

1st Proof: Take a sequence of step functions $f_n \to f$ a.e.

For each step function f_n, we may find a set A_n with $m(A_n) < \frac{\varepsilon}{2^n} \cdot \frac{1}{3}$ such that f_n is continuous on $A \setminus A_n$.

By Egorov, there is a set A_ε with $m(A_\varepsilon) < \frac{\varepsilon}{3}$ s.t. $f_n \to f$ uniformly on $A \setminus A_\varepsilon$.

Let $\hat{A} = A \setminus (\bigcup_{n=1}^\infty A_n \cup A_\varepsilon)$. ~~Then each f_n is continuous on $\hat{A}$, and~~

Then $\hat{A} \in \mathcal{L}$ and $m(A \setminus \hat{A}) < \frac{2}{3}\varepsilon$.

So $\exists$ closed set $F \subset A \setminus \hat{A}$ with $m((A \setminus \hat{A}) \setminus F) < \frac{\varepsilon}{3}$.

Then F is closed, $m(A \setminus F) < \varepsilon$. Moreover, each f_n is continuous on F, and $f_n \to f$ uniformly on $F \Rightarrow f$ is continuous on F. □

2^{nd} Proof. (Without using Egorov).

- First assume f is a simple function, i.e.

$$f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x), \quad \text{where } A_i \in \mathcal{L} \text{ disjoint}, \text{ and } A = \bigcup_{i=1}^{n} A_i.$$

For each A_i, choose a closed set $F_i \subset A_i$ s.t. $m(A_i \setminus F_i) < \frac{\varepsilon}{n}$.

Then the set $F := \bigcup_{i=1}^{n} F_i$ is closed and $m(A \setminus F) = \sum_{i=1}^{n} m(A_i \setminus F_i) < \varepsilon$.

Moreover, f is continuous on each F_i (as a constant function) $\Rightarrow$ f is continuous on F.
(since $F_i \cap F_j = \phi$). $\boxed{\text{should think about this!}}$

- Next suppose f is bounded.

Then $\exists$ simple functions $f_1, f_2, \cdots$ s.t. $f_n \to f$ uniformly.

For each n, choose closed set F_n s.t. f_n is continuous on F_n, and $m(A \setminus F_n) < \frac{\varepsilon}{2^n}$.
(by step 1).

Now let $F := \bigcap_{n=1}^{\infty} F_n$. Then $F \subset A$ is closed, and

$$m(A \setminus F) \le \sum_{n=1}^{\infty} m(A \setminus F_n) < \varepsilon.$$

Moreover, $f_n \to f$ uniformly on $F \Rightarrow f$ is continuous on F.

- Finally suppose f is any a.e. finite and measurable function.

Let $g(x) = \dfrac{f(x)}{1 + |f(x)|}$.

Then $g(x)$ is bounded and measurable.

So by Step 2, $\exists$ closed set $F \subset A$ with $m(A \setminus F) < \varepsilon$, s.t. g is continuous on F.

It follows that $f(x) = \dfrac{g(x)}{1 - |g(x)|}$ is continuous on F. $\square$

Cor.: Let f be a a.e. finite and measurable function defined on $A \in \mathcal{L}$. Then $\forall \varepsilon > 0$, $\exists$ a continuous function g on $\mathbb{R}^n$ s.t.

$$m(\{x \in A : f(x) \neq g(x)\}) < \varepsilon.$$

~~One sees~~: This follows from Lusin's theorem and

<u>Tietze Extension Theorem.</u> Suppose $F \subset \mathbb{R}^n$ is closed, and f is a continuous function defined on F. Then $\exists$ continuous function g on $\mathbb{R}^n$ s.t.

$$g(x) = f(x) \quad \text{on } F.$$

Moreover, if $|f(x)| \le M$ on F, then $|g(x)| \le M$ on $\mathbb{R}^n$.

~~Proof of Cor.~~

Cor.: Let f be a a.e. finite and ~~continuous~~ measurable function defined on $A \in \mathcal{L}$. Then $\exists$ continuous functions f_k defined on $\mathbb{R}^n$ s.t. $f_k \to f$ a.e. on A.

<u>Proof</u>: By previous corollary, we can find a sequence of continuous functions on $\mathbb{R}^n$ s.t. $f_n \to f$ in measure in A. So by Riesz thm, one can pick a subsequence f_{n_k} from f_n s.t. $f_{n_k} \to f$ a.e. in A. $\square$

Proof of Tietze Extension Theorem

- First assume $|f(x)| \le M$. We define

$F_1 = \{x \in F : \frac{M}{3} \le f(x) \le M\}$, $F_2 = \{x \in F : -M \le f(x) \le -\frac{M}{3}\}$, $F_3 = \{x \in F : -\frac{M}{3} < f(x) < \frac{M}{3}\}$.

Let
$$g_1(x) = \frac{M}{3} \frac{d(x, F_2) - d(x, F_1)}{d(x, F_2) + d(x, F_1)}$$
← continuous on $\mathbb{R}^n$. (denominar > 0 since $F_1 \cap F_2 = \phi$ and F_1, F_2 are closed.)

Then $|g_1(x)| \le \frac{M}{3}$ $(\forall x \in \mathbb{R}^n)$ and $|f(x) - g_1(x)| \le \frac{2}{3} M$. $(\forall x \in F)$ ← (to see this, consider F_1, F_2, F_3 separately.)

Then apply the same construction to $f(x) - g_1(x)$, to construct a continuous function g_2 on $\mathbb{R}^n$ s.t.
$$|g_2(x)| \le \frac{1}{3} \cdot \frac{2}{3} M \ (x \in \mathbb{R}^n) \quad \text{and} \quad |(f(x) - g_1(x)) - g_2(x)| \le \frac{2}{3} \cdot \frac{2}{3} M. \ (x \in F)$$

Continue this process, one get a sequence of continuous functions
$$g_1, g_2, g_3, \ldots$$
on $\mathbb{R}^n$ s.t.
$$|g_k(x)| \le \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{k-1} M \ (x \in \mathbb{R}^n), \quad \left|f(x) - \sum_{j=1}^{k} g_j(x)\right| \le \left(\frac{2}{3}\right)^k M. \ (x \in F)$$

It follow that $g(x) = \sum_{k=1}^{\infty} g_k(x)$ is continuous on $\mathbb{R}^n$, (by uniform convergence), and $g(x) = f(x)$, $\forall x \in F$.

Moreover,
$$|g(x)| \le \sum_{k=1}^{\infty} |g_k(x)| \le \sum_{k=1}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{k-1} M = \frac{M}{3} \cdot \frac{1}{1 - \frac{2}{3}} = M.$$

- If f is unbounded, then consider
$$\hat{f}(x) = \frac{f(x)}{1 + |f(x)|}. \quad \rightsquigarrow \text{continuous on } F.$$

By part 1, $\exists$ continuous function $\hat{g}(x)$ on $\mathbb{R}^n$ s.t. $\hat{g} = \hat{f}$ on F, and

It follows that $g = \frac{\hat{g}(x)}{1 - |\hat{g}(x)|}$ is continuous on $\mathbb{R}^n$, and $g = f$ on F. $\square$

[One should be a little bit careful.

f is continuous on F $\Rightarrow$ f is continuous on $F \cap \overline{B(0,R)}$, $\forall R$

$\Rightarrow$ f is bounded on $F \cap \overline{B(0,R)}$, $\forall R$

$\Rightarrow$ $|\hat{f}| \le C_R < 1$ on $F \cap \overline{B(0,R)}$

$\Rightarrow$ $|\hat{g}| \le C_R < 1$ ------

$\Rightarrow$ g is continuous ------]

[one can use $\tilde{f}(x) = \arctan f(x)$. $\sim$ simpler.]