Real Analysis

Lecture 9
03/26/2018

Last time.

- Any measurable function f is the limit of a sequence of simple functions f_n's:
  - If $f \geq 0$, then $\exists\, 0 \leq f_1 \leq f_2 \leq \cdots \leq f$ s.t. $\lim_{n\to\infty} f_n = f$
  - In general, $\exists\, 0 \leq |f_1| \leq |f_2| \leq \cdots \leq |f|$ s.t. $\lim_{n\to\infty} f_n = f$
  - In f is bounded, then $f_n \to f$ uniformly.
- Lusin: $\forall \varepsilon > 0$, $\exists F \subset A$ closed, with $m(A \setminus F) < \varepsilon$, s.t. f is continuous on F
  - $\forall \varepsilon > 0$, $\exists$ continuous function g on $\mathbb{R}^n$ s.t. $m(\{x \in A: f(x) \neq g(x)\}) < \varepsilon$.
  - $\exists$ a sequence of continuous functions $f_1, f_2, \cdots$ on $\mathbb{R}^n$ s.t. $f_n \to f$ a.e. on A

Today. Lebesgue's integral of non-negative functions

~~0. Integration of simple functions~~ 1. Some explanation of Lebesgue's integration

- Idea: to define $\int_A f(x)\,dx$, one first define $\int_A f_n(x)\,dx$, where f_n is a simple function. then "approximate" $\int_A f(x)\,dx$ via $\int_A f_n(x)\,dx$.

  Convergence issue: If $f_n \to f$, do we really have "$\int_A f_n(x)\,dx \to \int_A f(x)\,dx$"?

- To get a clearer idea of how to solve this issue, let's look at infinite sums

  "discrete version of integrals"

  Consider an infinite summation $\sum_{n=1}^{\infty} a_n$.

  There are two different (but related) theory of the sum.

  Note: As "integrals", we don't care the "order of summation!

  (1) "non-negative theory": If $a_n \geq 0$ for all n, then
  $$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n \in [0, +\infty].$$
  The limit always exists, and could be infinity.

  Moreover, one can change the order
  $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)} \quad (\text{where } \sigma: \mathbb{N} \to \mathbb{N} \text{ is a bijection})$$
  $$= \sup_{S \subset \mathbb{N},\, S \text{ finite}} \sum_{n\in S} a_n.$$

  (2) "absolute summable theory": If $a_n \in \mathbb{R}$ (or $a_n \in \mathbb{C}$), to make the summation $\sum_{n=1}^{\infty} a_n$ independant of order, one need to pose the "absolute summable assumption"
  $$\sum_{n=1}^{\infty} |a_n| < \infty.$$

  [Think about this: why do we want order-independence?]

  In this case, one can define
  $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- \quad \left(\text{for } a_n \in \mathbb{R}: \ a_n^+ = \max(a_n, 0),\ a_n^- = \max(-a_n, 0)\right)$$

  [and if $a_n \in \mathbb{C}$: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} Re(a_n) + i \sum_{n=1}^{\infty} Im(a_n)$.]

- In analogue to infinite summations, we will see two (overlapping) types of integration theory: <u>non-negative theory</u> and <u>absolutely integrable theory</u>.

  [And: we will use the 1^{st} one to develop the 2^{nd} one.]

  We will prove different convergence theorems.

  [Recall from lecture 1: one crucial advantage of Lebesgue's theory is that one have convergence theorems under very mild assumptions!]

  - Nonnegative theory
    - $\rightsquigarrow$ Levi's monotone convergence theorem
    - $\rightsquigarrow$ Fatou lemma
  - Absolute integrable theory
    - $\rightsquigarrow$ Lebesgue's dominated convergence theorem.

  [These convergence thms make Lebesgue's theory particularly useful in applications!]

2. Integration of simple functions

- For any measurable set $A \in \mathcal{L}$, it is reasonable to define
$$\int_{\mathbb{R}^d} \chi_A(x)\,dx := \int_A 1\,dx := m(A).$$
  Using this and "linearity" of integration, one can define

  Def: For any simple function $f(x) = \sum_{n=1}^{N} a_n \chi_{A_n}(x)$, one define
$$\int_{\mathbb{R}^d} f(x)\,dx = \sum_{n=1}^{\infty} a_n\, m(A_n).$$

  One should check that this is well-defined. In other words, if $\sum a_n \chi_{A_n} = \sum \hat{a}_n \chi_{\hat{A}_n}$, then $\sum a_n m(A_n) = \sum \hat{a}_n m(\hat{A}_n)$.

  [More generally, for any measurable set $\hat{A} \in \mathcal{L}$, we define
$$\int_{\hat{A}} f(x)\,dx = \int_{\mathbb{R}^d} \chi_{\hat{A}} \cdot f(x)\,dx = \sum_{n=1}^{\infty} a_n\, m(A_n \cap \hat{A})$$
  since $\chi_{\hat{A}} \cdot f(x) = \sum_{n=1}^{\infty} a_n \chi_{A_n \cap \hat{A}}(x)$.

  As a result, it is enough to define $\int_{\mathbb{R}^d} f\,dx$: If f is only defined on A, we can extend f to a function on $\mathbb{R}^d$ by defining $f = 0$ on $\mathbb{R}^d \setminus A$.]

  Example: $\int_{\mathbb{R}} \chi_{\mathbb{Q}}(x)\,dx = 0$.

- Let's list basic properties of integration of non-negative simple functions.

  Prop: Let f, g be non-negative simple functions.

  (1) For any $c_1, c_2 \geq 0$, one has
$$\int_{\mathbb{R}^d} (c_1 f + c_2 g)\,dx = c_1 \int_{\mathbb{R}^d} f(x)\,dx + c_2 \int_{\mathbb{R}^d} g(x)\,dx.$$

  (2) If $f \leq g$, then $\int_{\mathbb{R}^d} f(x)\,dx \leq \int_{\mathbb{R}^d} g(x)\,dx$.

  (3) $\int_{\mathbb{R}^d} f(x)\,dx < \infty \Leftrightarrow f$ is finite almost everywhere, and ~~supp f~~ $m(\mathrm{supp} f) < \infty$.

  where $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$. [Note: We don't take closure here.]

  (4) $\int_{\mathbb{R}^d} f(x)\,dx = 0 \Leftrightarrow f = 0$ a.e.

  (5) For any $y \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} f(x+y)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$.

Proof: (1). Let $f(x)=\sum_{i=1}^n a_i \chi_{A_i}(x)$, $g(x)=\sum_{j=1}^m b_j \chi_{B_j}(x)$, where $\bigcup_{i=1}^n A_i = \bigcup_{j=1}^m B_j = \mathbb{R}^d$, then

$$c_1 f + c_2 g = \sum_{i=1}^n \sum_{j=1}^m (c_1 a_i + c_2 b_j)\chi_{A_i\cap B_j} \quad \leftarrow \left(\bigcup_{i,j} A_i\cap B_j = \mathbb{R}^d,\ \text{pairwise disjoint}\right)$$

is a simple function, and

$$\begin{aligned}
\int_{\mathbb{R}^d}(c_1 f + c_2 g)\,dx &= \sum_{i=1}^n \sum_{j=1}^m (c_1 a_i + c_2 b_j)\, m(A_i\cap B_j) \\
&= c_1 \sum_{i=1}^n \sum_{j=1}^m a_i\, m(A_i\cap B_j) + c_2 \sum_{j=1}^m \sum_{i=1}^n b_j\, m(A_i\cap B_j) \\
&= c_1 \sum_{i=1}^n a_i\, m(A_i) + c_2 \sum_{j=1}^m b_j\, m(B_j) \\
&= c_1 \int_{\mathbb{R}^d} f\,dx + c_2 \int_{\mathbb{R}^d} g\,dx .
\end{aligned}$$

(2). If $f\le g$, then $g-f$ is a non-negative simple function, since

$$g-f = \sum_{i=1}^n \sum_{j=1}^m (b_j - a_i)\chi_{A_i\cap B_j} \quad \leftarrow (\text{simple}).$$

So $b_j - a_i \ge 0$ if $A_i\cap B_j \neq \phi$.

It follows by definition that

$$\int_{\mathbb{R}^d}(g-f)\,dx \ge 0 .$$

So

$$\int_{\mathbb{R}^d} f\,dx \le \int_{\mathbb{R}^d} f\,dx + \int_{\mathbb{R}^d}(g-f)\,dx = \int_{\mathbb{R}^d} g\,dx$$

(3). • If f is finite a.e., and $m(\operatorname{supp} f)<\infty$, then

$$m(\{x: f(x)=+\infty\})=0$$

and $a := \sup\{f(x): f(x)<+\infty\} < +\infty$. (because f only take finitely many different values)

It follows

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)\,dx &= \sum_{i=1}^n a_i\, m(A_i) \\
&\le a \cdot \sum_{0<a_i<+\infty} m(A_i) \\
&= a\cdot m(\operatorname{supp} f) < +\infty .
\end{aligned}$$

• Conversely, if f is not finite a.e., i.e. $m(\{x: f(x)=+\infty\})>0$, then

$$\int_{\mathbb{R}^d} f\,dx \ge +\infty \cdot m(\{x: f(x)=+\infty\}) = +\infty ;$$

if $m(\operatorname{supp} f)=+\infty$, then let $b=\inf\{f(x): f(x)>0\}>0$, we get

$$\int_{\mathbb{R}^d} f\,dx \ge b\cdot m(\operatorname{supp} f) = +\infty .$$

(think about this).

(4) If $f=0$ a.e., then $A=\{x: f(x)\ne 0\}=0$. So $m(A_i)=0$ for $a_i\ne 0$.

It follows $\int_{\mathbb{R}^d} f(x)\,dx = \sum_{i=1}^n a_i\, m(A_i) = 0$.

If $\int_{\mathbb{R}^d} f(x)\,dx = \sum_{i=1}^n a_i\, m(A_i) = 0$, then $\forall a_i>0$, one has $m(A_i)=0$.

So $A=\{x: f(x)\neq 0\} = \{x: f(x)=a_i>0\} = \bigcup_{a_i>0} A_i$ and $m(A)=\sum_{a_i>0} m(A_i)=0$. So $f=0$ a.e. $\square$

(5). $f(x+y)=\sum_{i=1}^n a_i \chi_{A_i - y}(x)$ is simple, and

$$\int_{\mathbb{R}^d} f(x+y)\,dx = \sum_{i=1}^n a_i\, m(A_i - y) = \sum_{i=1}^n a_i\, m(A_i) = \int_{\mathbb{R}^d} f(x)\,dx.$$

We could define $\int_{\mathbb{R}^d} f(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx$, where $f_n \nearrow f$, f_n are simple.
Then we need to check the well definedness, which is not obvious.
Moreover, we can't use this to define lower/upper integrals.

3. Integration of non-negative functions

- Now let f be any non-negative measurable function.

As we have seen, one can approximate f by a sequence of non-negative, increasing simple functions.
This motivates us to define

Def.: Suppose f is non-negative and measurable. We define the Lebesgue integral of f to be

$$\int_{\mathbb{R}^d} f(x)\,dx = \sup\left\{ \int_{\mathbb{R}^d} g(x)dx : 0 \le g \le f,\ g \text{ is simple (and measurable)} \right\} \quad (*)$$

Rmk:
- Similarly for any measurable set A, one can define

$$\int_A f(x)dx = \int_{\mathbb{R}^n} f(x)\chi_A(x)\,dx.$$

- If f is not measurable, one can still define the lower Lebesgue integral of f via $(*)$:

$$\underline{\int_{\mathbb{R}^d}} f(x)dx = \sup\left\{ \int_{\mathbb{R}^d} g(x)dx : 0 \le g \le f,\ g \text{ is simple (and measurable)} \right\}.$$

and define the upper Lebesgue integral of f similarly

$$\overline{\int_{\mathbb{R}^d}} f(x)dx = \inf\left\{ \int_{\mathbb{R}^d} g(x)dx : f \le g,\ g \text{ is simple (and measurable)} \right\}.$$

(If f is measurable AND ~~of fin~~ bounded AND $m(\operatorname{supp} f) < +\infty$, then the lower and upper Lebesgue integrals agree.)

- Again let's list basic properties of integration of non-negative measurable functions.

Prop.: Let f, g be non-negative and measurable functions. Then

(1) If $f \le g$, then $\int_{\mathbb{R}^d} f(x)dx \le \int_{\mathbb{R}^d} g(x)dx$.

(2) For any $c_1, c_2 \ge 0$, one has

$$\int_{\mathbb{R}^d} (c_1 f + c_2 g)dx = c_1 \int_{\mathbb{R}^d} f(x)dx + c_2 \int_{\mathbb{R}^d} g(x)dx.$$

(3) $\int_{\mathbb{R}^d} f(x)dx = 0 \iff f(x) = 0$ a.e. $x \in \mathbb{R}^d$.

(4) [Translation invariance] For any $y \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} f(x+y)dx = \int_{\mathbb{R}^d} f(x)dx$.

(5) [Markov's inequality] For any $0 < \lambda < \infty$, one has

$$m(\{x \in \mathbb{R}^d : f(x) \ge \lambda\}) \le \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x)dx.$$

Proof. (1) If h is simple, $0 \le h \le f$, then $0 \le h \le g$. It follows

$$\int_{\mathbb{R}^d} f(x)\,dx = \sup\left\{\int_{\mathbb{R}^d} h(x)\,dx : 0 \le h \le f,\ h \text{ is simple}\right\} \le \int_{\mathbb{R}^d} g(x)\,dx.$$

(2). We first prove $\int_{\mathbb{R}^d} cf(x)\,dx = c\int_{\mathbb{R}^d} f(x)\,dx$ for any $c \ge 0$.

This is true because $\int_{\mathbb{R}^d} ch(x)\,dx = c\int_{\mathbb{R}^d} h(x)\,dx$ for simple function h.

It follows

$$\begin{aligned}\int_{\mathbb{R}^d} cf(x)\,dx &= \sup\left\{\int_{\mathbb{R}^d} ch(x)\,dx : 0 \le ch \le cf,\ h \text{ is simple}\right\}\\ &= c\cdot \sup\left\{\int_{\mathbb{R}^d} h(x)\,dx : 0 \le h \le f,\ h \text{ is simple}\right\}\\ &= c\int_{\mathbb{R}^d} f(x)\,dx.\end{aligned}$$

It remains to prove $\int_{\mathbb{R}^d}(f+g)\,dx = \int_{\mathbb{R}^d} f\,dx + \int_{\mathbb{R}^d} g\,dx$.

We will postpone the proof for a while. [We will use Levi's monotone convergence thm to prove additivity. Try: can you prove this additivity without using convergence thm?]

(3). If $f = 0$ a.e., and $0 \le h \le f$, then $h = 0$ a.e., so $\int_{\mathbb{R}^n} h(x)\,dx = 0$.
simple

It follows that

$$\int_{\mathbb{R}^d} f(x)\,dx = \sup\left\{\int_{\mathbb{R}^d} h(x)\,dx : 0 \le h \le f,\ h \text{ simple}\right\} = 0.$$

Conversely, according to (5) (the Markov inequality), for any $k > 0$,

$$m\left(\left\{x \in \mathbb{R}^d : f(x) \ge \tfrac{1}{k}\right\}\right) \le k\int_{\mathbb{R}^d} f(x)\,dx = 0.$$

So $m(\{x : f(x) \ne 0\}) = m\left(\bigcup_{k=1}^{\infty}\{x : f(x) \ge \tfrac{1}{k}\}\right) \le \sum_{k=1}^{\infty} m\left(\{x : f(x) \ge \tfrac{1}{k}\}\right) = 0.$

In other words, $f = 0$ a.e.

(4) If h is simple and $0 \le h \le f$, $\forall x$

then $h(x+y)$ is simple as a function of x, and $0 \le h(x+y) \le f(x+y)$, $\forall x$.

~~Conversely, if $h(x+y)$ is simple as a functi~~ It follows

$$\int_{\mathbb{R}^d} f(x+y)\,dx \le \sup\left\{\int_{\mathbb{R}^d} h(x+y)\,dx : 0 \le h \le f,\ h \text{ is simple}\right\} = \int_{\mathbb{R}^d} f(x)\,dx.$$

Similarly $\int_{\mathbb{R}^d} f(x)\,dx \le \int_{\mathbb{R}^d} f(x+y)\,dx$. So $\int_{\mathbb{R}^d} f(x+y)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$.

(5). By definition we have

$$\lambda\cdot\chi_{\{x \in \mathbb{R}^d : f(x) \ge \lambda\}} \le f(x), \quad \forall x.$$

So by (1), one has

$$\lambda\cdot m(\{x \in \mathbb{R}^d : f(x) \ge \lambda\}) = \int_{\mathbb{R}^d} \lambda\chi_{\{x \in \mathbb{R}^d : f(x) \ge \lambda\}}\,dx \le \int_{\mathbb{R}^d} f(x)\,dx. \qquad \square$$

- Note that in the previous propositions, the integral could be $+\infty$.

<u>Def.</u> A non-negative measurable function f is said to be <u>integrable</u>, if
$$\int_{\mathbb{R}^d} f(x)\,dx < \infty.$$

We have seen that for a non-negative <u>simple</u> function f, it is integrable if and only if f is a.e. finite <u>AND</u> $m(\operatorname{supp} f) < +\infty$.

For general non-negative ~~functions~~ measurable functions, one can easily construct integrable function f with $m(\operatorname{supp} f) = +\infty$.

In the proof we have seen that $m(\operatorname{supp} f) < +\infty$ for integrable non-negative simple function f because f takes only finitely many distinct values. Of course this is no longer true for measurable functions.

For example, one just take $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \chi_{[n,n+1)}$. (<u>Check</u>: $\int_{\mathbb{R}} f(x)\,dx = \frac{\pi^2}{6}$.)

However, one still have

<u>Prop.</u> If f is integrable, then f is a.e. finite.

<u>Proof</u>: This also follows from Markov's inequality:
$$m(\{x : f(x) = +\infty\}) = m\Big(\bigcap_{k=1}^{\infty} \{x : f(x) \geq k\}\Big) \leq m(\{x : f(x) \geq k\}) \leq \frac{1}{k}\int_{\mathbb{R}^d} f(x)\,dx.$$

Letting $k \to \infty$, we get
$$m(\{x : f(x) = +\infty\}) = 0. \qquad \square$$