Real Analysis

Lecture 10
03/29/2018

Last time

- Simple function $f(x) = \sum_{n=1}^{N} a_n \chi_{A_n} \rightsquigarrow \int_{\mathbb{R}^d} f(x)dx = \sum_{n=1}^{N} a_n m(A_n)$

  Non-negative measurable function $f \rightsquigarrow \int_{\mathbb{R}^d} f(x)dx = \sup\left\{\int_{\mathbb{R}^n} g(x)dx : 0 \le g \le f,\ g \text{ is simple}\right\}$

  Basic properties
  - $\int_{\mathbb{R}^d} (c_1 f_1 + c_2 f_2)\,dx = c_1 \int_{\mathbb{R}^d} f_1(x)dx + c_2 \int_{\mathbb{R}^d} f_2(x)dx$
  - $f_1 \le f_2 \Rightarrow \int_{\mathbb{R}^d} f_1(x)dx \le \int_{\mathbb{R}^d} f_2(x)dx$
  - $\int_{\mathbb{R}^d} f(x)dx = 0 \Leftrightarrow f = 0$ a.e.
  - $\int_{\mathbb{R}^d} f(x+y)dx = \int_{\mathbb{R}^d} f(x)dx$.

- We say f is integrable if $\int_{\mathbb{R}^d} f(x)dx < +\infty$
  - A simple function f in integrable $\Leftrightarrow$ f is finite a.e., and $m(\operatorname{supp} f) < +\infty$
  - A non-negative measurable function f is integrable $\Rightarrow$ f is finite a.e.

Today: Convergence of integrals

1. Convergence in non-negative theory

- Recall: In Lecture 4, we proved monotone convergence of measurable sets:

$$A_1 \subset A_2 \subset A_3 \subset \cdots \Rightarrow m\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

~~As a consequence of we let~~

We can translate this to properties of characteristic function χ_{A_n}'s:

$$\chi_{A_1} \le \chi_{A_2} \le \chi_{A_3} \le \cdots \Rightarrow \int_{\mathbb{R}^d} \lim_{n\to\infty} \chi_{A_n}\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \chi_{A_n}\,dx.$$

It turns out that this is true in general:

Thm. (Levi's monotone convergence theorem)

Let f_n be a monotonely increasing sequence of nonnegative measurable functions.

$$0 \le f_1 \le f_2 \le f_3 \le \cdots \le f_n \le \cdots$$

Then

$$\int_{\mathbb{R}^d} \lim_{n\to\infty} f_n\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx$$

[Note: Both limits exist!]

Proof.
- Denote $f(x) := \lim_{n\to\infty} f_n(x)$ $(= \sup_n f_n(x))$. Then $f_n \le f \Rightarrow \int_{\mathbb{R}^d} f_n(x)dx \le \int_{\mathbb{R}^d} f(x)dx$

  So we get $\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)dx \le \int_{\mathbb{R}^d} f(x)dx$.

- To prove the converse, let's apply our favorite ε-trick.

  Take any simple function $0 \le h \le f$, and define $A_n = \{x : f_n(x) \ge (1-\varepsilon)h(x)\}$.

  Note that $f_n \le f_{n+1} \Rightarrow A_n \subseteq A_{n+1}$. So we get an increasing sequence of sets

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n \subset \cdots$$

  The fact $h \le f = \lim_{n\to\infty} f_n$ implies $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}^d$.

Write $h(x) = \sum a_i \chi_{B_i}$.

It follows that

$$\lim_{n\to\infty} \int_{A_n} \underbrace{h(x)}_{\sum a_i \chi_{B_i}} dx = \lim_{n\to\infty} \sum_{i=1}^{N} a_i \, m(B_i \cap A_n)$$

$$= \sum_{i=1}^{N} a_i \, m\Big(B_i \cap \bigcup_{n=1}^{\infty} A_n\Big)$$

$$= \int_{\bigcup_{n=1}^{\infty} A_n} h(x)\,dx = \int_{\mathbb{R}^d} h(x)\,dx$$

$$\Rightarrow \int_{\mathbb{R}^d} f_n(x)\,dx \geq (1-\varepsilon)\int_{A_n} h(x)\,dx \cancel{\geq (1-\varepsilon)\int_{A_n} h(x)\,dx} \to \boxed{(1-\varepsilon)\int_{\mathbb{R}^d} h(x)\,dx}$$

$$\Rightarrow \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,dx \geq (1-\varepsilon)\int_{\mathbb{R}^d} h(x)\,dx$$

Taking supremum over h, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,dx \geq (1-\varepsilon)\int_{\mathbb{R}^d} f(x)\,dx.$$

Letting $\varepsilon \to 0$, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,dx \geq \int_{\mathbb{R}^d} f(x)\,dx. \qquad \square$$

Here is how you get the proof:

WANT: $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,dx \geq \int_{\mathbb{R}^d} f(x)\,dx$

$\xLeftarrow{\varepsilon\text{-trick}}$ $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,dx \geq (1-\varepsilon)\int_{\mathbb{R}^d} f(x)\,dx$

$\xLeftarrow{\text{def.}}$ $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,dx \geq (1-\varepsilon)\int_{\mathbb{R}^d} h(x)\,dx$

where $0 \leq h \leq f$, h is simple.

But we only have a weaker inequality

$$\int_{\mathbb{R}^d} f_n(x)\,dx \geq \int_{f_n \geq (1-\varepsilon)h} (1-\varepsilon)h\,dx$$

So we need to introduce $A_n := \{x : f_n \geq (1-\varepsilon)h\}$ and try to prove $\int_{A_n} h\,dx \to \int_{\mathbb{R}^d} h(x)\,dx$

So we need moniotonicity of A_n and need $\mathbb{R}^d = \bigcup_{n=1}^{\infty} A_n$.

Cor. (Integration by terms). For any sequence of non-negative measurable functions f_n, one has

$$\int_{\mathbb{R}^d} \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} f_n(x)\,dx.$$

~~[In particular, $\int_{\mathbb{R}^d}(f+g)\,dx = \int_{\mathbb{R}^d} f\,dx + \int_{\mathbb{R}^d} g\,dx$, So we complete the proof of linearity.]~~

Proof: $\sum_{n=1}^{N} f_n(x) =: F_N(x)$ is an increasing sequence. So by monotone convergence,

$$\int_{\mathbb{R}^d} \sum_{n=1}^{\infty} f_n(x)\,dx = \int_{\mathbb{R}^d} \lim_{N\to\infty} F_N(x)\,dx = \lim_{N\to\infty} \int_{\mathbb{R}^d} F_N(x)\,dx = \lim_{N\to\infty} \sum_{n=1}^{N} \int_{\mathbb{R}^d} f_n(x)\,dx$$

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} f_n(x)\,dx. \qquad \square$$

Cor. (linearity) $\int_{\mathbb{R}^d} (f+g)\,dx = \int_{\mathbb{R}^d} f(x)\,dx + \int_{\mathbb{R}^d} g(x)\,dx$.

[As a consequence, $\int_{\mathbb{R}^d} \sum_{n=1}^{N} f_n(x)\,dx = \sum_{n=1}^{N} \int_{\mathbb{R}^d} f_n(x)\,dx$]

Proof: Pick monotone sequence of simple functions, $\lim_{n\to\infty} \varphi_n = f$, $\lim_{n\to\infty} \psi_n = g$, s.t.

$$0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_n \leq \cdots \leq f, \qquad 0 \leq \psi_1 \leq \psi_2 \leq \cdots \leq \psi_n \leq \cdots \leq g.$$

Then $0 \leq \varphi_1 + \psi_1 \leq \varphi_2 + \psi_2 \leq \cdots \leq \varphi_n + \psi_n \leq \cdots \leq f+g$, $\lim_{n\to\infty}(\varphi_n + \psi_n) = f+g$.

So by monotone convergence theorem,

$$\int_{\mathbb{R}^d} (f+g)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} (\varphi_n + \psi_n)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n\,dx + \lim_{n\to\infty} \int_{\mathbb{R}^d} \psi_n\,dx$$

$$= \int_{\mathbb{R}^d} \lim_{n\to\infty} \varphi_n\,dx + \int_{\mathbb{R}^d} \lim_{n\to\infty} \psi_n\,dx = \int_{\mathbb{R}^d} f(x)\,dx + \int_{\mathbb{R}^d} g(x)\,dx. \qquad \square$$

(**Cor.** For non-negative f, we can define $\int_{\mathbb{R}^d} f(x)\,dx = \lim \int_{\mathbb{R}^d} f_n(x)\,dx$, where $0 \leq f_1 \leq f_2 \leq \cdots \nearrow f$, f_n simple.)

Cor.: Suppose $A_n \in \mathcal{L}$, $A_i \cap A_j = \phi$ $(i \neq j)$. Then for any non-negative measurable function f,

$$\int_{\bigcup_{n=1}^{\infty} A_n} f(x)\,dx = \sum_{n=1}^{\infty} \int_{A_n} f(x)\,dx.$$

So: $\mu(A) := \int_A f\,dx$ defines a measure $\mu : \mathcal{L} \to R$.

Proof.
$$\sum_{n=1}^{\infty}\int_{A_n} f(x)\,dx = \sum_{n=1}^{\infty}\int_{\mathbb{R}^d} f(x)\chi_{A_n}(x)\,dx = \int_{\mathbb{R}^d}\sum_{n=1}^{\infty} f(x)\chi_{A_n}(x)\,dx$$
$$= \int_{\mathbb{R}^d} f(x)\cdot\chi_{\bigcup_{n=1}^{\infty} A_n}(x)\,dx = \int_{\bigcup_{n=1}^{\infty} A_n} f(x)\,dx. \quad \square$$

- In general, if we are given a sequence of non-negative measurable functions, f_n, it is possible that they don't converge. In this case we have

Thm (Fatou lemma) For any sequence of non-negative measurable functions f_n,
$$\int_{\mathbb{R}^n} \underbrace{\liminf_{n\to\infty} f_n}_{\text{measurable!}}\,dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^n} f_n(x)\,dx.$$

Proof. For every $j \ge k$, we have $\inf_{n\ge k} f_n \le f_j$. So we get
$$\int_{\mathbb{R}^d}\inf_{n\ge k} f_n\,dx \le \inf_{j\ge k}\int_{\mathbb{R}^d} f_j(x)\,dx$$

Note: Let $f_n(x) = \begin{cases} n, & 0<x<\frac{1}{n} \\ 0, & x\le 0 \text{ or } \frac{1}{n}\le x \end{cases}$. Then $\int_{\mathbb{R}} \lim f_n = 0 < 1 = \lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,dx$.

Now the sequence $g_k := \inf_{n\ge k} f_n$ is a monotone increasing sequence of non-negative measurable functions, and $\lim_{k\to\infty} g_k(x) = \liminf_{n\to\infty} f_n$. So by monotone convergence theorem,
$$\int_{\mathbb{R}^d}\liminf_{n\to\infty} f_n\,dx = \int_{\mathbb{R}^d}\lim_{k\to\infty} g_k(x)\,dx = \lim_{k\to\infty}\int_{\mathbb{R}^d} g_k(x)\,dx$$
$$\le \lim_{k\to\infty}\inf_{j\ge k}\int_{\mathbb{R}^d} f_j(x)\,dx = \liminf_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,dx. \quad \square$$

- Using additivity, one can also prove the following description of Lebesgue integral that we introduced in Lec 1:

Prop: Let f be an a.e. finite and measurable function on A, where $m(A) < +\infty$. Then
$$\int_A f(x)\,dx = \lim_{\delta\to 0}\sum_{k=0}^{\infty}(t_{k+1}-t_k)\,m(\{x : f(x)\ge t_k\}) \quad \left(= \int_0^{\infty} m(\{x : f(x)\ge t\})\,dt\right)$$
where $0 = t_0 < t_1 < t_2 < \cdots \to \infty$ and $t_{k+1} - t_k < \delta$.

Proof: Let $A_k = \{x : f(x) \ge t_k\}$. Then
$$t_k\,m(A_k\setminus A_{k+1}) \le \int_{A_k\setminus A_{k+1}} f(x)\,dx \le t_{k+1}\,m(A_k\setminus A_{k+1}).$$
$$\Rightarrow \sum_{k=0}^{\infty} t_k(m(A_k) - m(A_{k+1})) \le \int_A f(x)\,dx \le \sum_{k=0}^{\infty} t_{k+1}(m(A_k) - m(A_{k+1})) = \sum_{k=0}^{\infty}(t_{k+1}-t_k)\,m(A_k)$$
$$\| \quad \sum_{k=0}^{\infty}(t_{k+1}-t_k)\,m(A_{k+1})$$
$$\| \quad \sum_{k=0}^{\infty}(t_{k+1}-t_k)(m(A_{k+1}) - m(A_k)) + \sum_{k=0}^{\infty}(t_{k+1}-t_k)\,m(A_k) \ge \sum_{k=0}^{\infty}(t_{k+1}-t_k)\,m(A_k) - \delta\cdot m(A).$$
$\square$