# Real Analysis

Lecture 11
04/09/2018

**Last time**: "Convergence in non-negative theory"

- **Levi's monotone convergence theorem**: $0 \le f_1 \le f_2 \le \cdots \le f_n \le \cdots$

$$\Rightarrow \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n(x)\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\, dx.$$

**Fatou's lemma**: For any non-negative measurable functions $\{f_n\}$, one has

$$\int_{\mathbb{R}^d} \liminf_{n\to\infty} f_n(x)\, dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\, dx.$$

- Consequences:
  - For non-negative measurable functions $\{f_n\}$, one has
  $$\int_{\mathbb{R}^d} \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} f_n(x)\, dx.$$
  - For $A_n \in \mathcal{L}$, $A_i \cap A_j = \phi$, ~~one has~~ and for any non-negative measurable function $f$, one has
  $$\int_{\bigcup_{n=1}^{\infty} A_n} f(x)\, dx = \sum_{n=1}^{\infty} \int_{A_n} f(x)\, dx.$$

**Today**: Lebesgue integral for measurable functions ("Absolutely integrable" theory)

## 1. Lebesgue integral for general measurable functions: definition

- Let $f$ be any measurable function defined on $\mathbb{R}^d$.
  [It is enough to assume that $f$ is a.e. defined on $\mathbb{R}^d$.]

**Recall**: $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$ are both non-negative measurable functions, and $f(x) = f^+(x) - f^-(x)$, $|f(x)| = f^+(x) + f^-(x)$.

We want to define $\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d} f^+(x)\,dx - \int_{\mathbb{R}^d} f^-(x)\,dx$, but this makes no sense if both integrals $\int_{\mathbb{R}^d} f^+(x)\,dx$ and $\int_{\mathbb{R}^d} f^-(x)\,dx$ are infinite.

**Def**:
- If at least one of $\int_{\mathbb{R}^d} f^+(x)\,dx$ and $\int_{\mathbb{R}^d} f^-(x)\,dx$ are finite, then we define
$$\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d} f^+(x)\,dx - \int_{\mathbb{R}^d} f^-(x)\,dx.$$
- We say a measurable function $f$ is <u>integrable</u> if both $\int_{\mathbb{R}^d} f^+(x)\,dx$ and $\int_{\mathbb{R}^d} f^-(x)\,dx$ are finite. ($\Rightarrow \int_{\mathbb{R}^d} f(x)\,dx$ is finite.)
- We denote the space of integrable functions on $\mathbb{R}^d$ by $L^1(\mathbb{R}^d)$

**Rmk**:
- If $f$ is defined on a measurable set $A$, one can perform the "zero extension"
$$\tilde{f}(x) = \begin{cases} f(x), & x \in A \\ 0, & x \notin A \end{cases} = f(x)\cdot\chi_A(x),$$ and define $$\int_A f(x)\,dx = \int_{\mathbb{R}^d} f(x)\,\chi_A(x)\,dx$$
if the later is well-defined.
- $L^1(A)$ = the space of integrable function on $A$.

**Rmk**: By PSet 5, Part 1, Problem 2, if $A_1 \subset A$ and $m(A_1)=0$, then $\int_A f\,dx = \int_{A\setminus A_1} f\,dx$.

• The following proposition justifies the terminology "absolute integrability"

**Prop.** A measurable function $f \in L^1(\mathbb{R}^d)$ if and only if $\int_{\mathbb{R}^d} |f(x)|\, dx < +\infty$.

**Proof.** • If $\int_{\mathbb{R}^d} |f|\, dx < +\infty$, then $\int_{\mathbb{R}^d} f^+ dx \le \int_{\mathbb{R}^d} |f|\, dx < +\infty$, $\int_{\mathbb{R}^d} f^- dx \le \int_{\mathbb{R}^d} |f|\, dx < +\infty$

$\Rightarrow f$ is integrable.

• If $f$ is integrable, then

$$\int_{\mathbb{R}^d} |f|\, dx = \int_{f>0} f^+ dx + \int_{f<0} f^- dx \le \int_{\mathbb{R}^d} f^+ dx + \int_{\mathbb{R}^d} f^- dx < +\infty. \quad \square$$

**Def.** We call the quantity

$$\|f\|_{L^1(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f(x)|\, dx$$

the $L^1$-norm of $f \in L^1(\mathbb{R}^d)$.

**Cor.** $L^1(\mathbb{R}^d)$ is a linear space.

**Prof.** If $f_1, f_2 \in L^1(\mathbb{R}^d)$, then $\forall c_1, c_2 \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} |c_1 f_1 + c_2 f_2|\, dx \le \int_{\mathbb{R}^d} (|c_1| \cdot |f_1| + |c_2| |f_2|)\, dx$$
$$\le |c_1| \underbrace{\int_{\mathbb{R}^d} |f_1| + |c_2| \int_{\mathbb{R}^d} |f_2|}_{<+\infty}$$

**Cor.** $f \in L^1(\mathbb{R}^d) \Rightarrow f$ is a.e. finite.

It is obvious that the $L^1$-norm satisfies

(1) $\|cf\|_{L^1(\mathbb{R}^d)} = |c| \cdot \|f\|_{L^1(\mathbb{R}^d)}$.

(2) $\|f_1 + f_2\|_{L^1(\mathbb{R}^d)} \le \|f_1\|_{L^1(\mathbb{R}^d)} + \|f_2\|_{L^1(\mathbb{R}^d)}$ ← triangle inequality.

(3) $\|f\|_{L^1(\mathbb{R}^d)} = 0 \iff f = 0$ a.e.

**Rmk:** Given any two integrable functions $f_1, f_2 \in L^1(\mathbb{R}^d)$, we can define their $L^1$-distance to be

$$d_{L^1}(f_1, f_2) := \|f_1 - f_2\|_{L^1(\mathbb{R}^d)}.$$

If we regard a.e. equal functions in $L^1(\mathbb{R}^d)$ as the same function, then $d_{L^1}$ is a "distance" on $L^1(\mathbb{R}^d)$, and we will show later that $L^1(\mathbb{R}^d)$ is complete w.r.t. $d_{L^1}$.

**Rmk.** For any $p \ge 1$, we can define

$$\|f\|_{L^p(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f(x)|^p\, dx,$$

and define the $L^p$-space

$$L^p(\mathbb{R}^d) = \{f : f \text{ is measurable, and } \|f\|_{L^p} < +\infty\}.$$

Again each $L^p(\mathbb{R}^d)$ is a linear space, and $\|\cdot\|_{L^p}$ gives a "norm" on it.

If $\|f_1\|_{L^p} < \infty$, $\|f_2\|_{L^p} < \infty$, then

$$\int |c_1 f_1 + c_2 f_2|^p \le \int 2^p (|c_1|^p |f_1|^p + |c_2|^p |f_2|^p) < \infty.$$

We will study the properties of $L^p(\mathbb{R}^d)$ later.

## 2. Basic properties of Lebesgue integrals.

**Prop.** Let $f_1, f_2 \in L^1(\mathbb{R}^d)$ be integrable functions. Then

(1) (linearity) $\int_{\mathbb{R}^d} (c_1 f_1 + c_2 f_2)\, dx = c_1 \int_{\mathbb{R}^d} f_1\, dx + c_2 \int_{\mathbb{R}^d} f_2\, dx, \quad \forall c_1, c_2 \in \mathbb{R}$.

(2) (monotonicity) If $f_1 \le f_2$, then $\int_{\mathbb{R}^d} f_1\, dx \le \int_{\mathbb{R}^d} f_2\, dx$.

(3) (translation-invariance) For any $y \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} f(x+y)\, dx = \int_{\mathbb{R}^d} f(x)\, dx$.

**Proof:** (1). It is enough to prove (a) $\int_{\mathbb{R}^d} cf\, dx = c \int_{\mathbb{R}^d} f\, dx$, $\forall c \in \mathbb{R}$.

(b) $\int_{\mathbb{R}^d} (f_1 + f_2)\, dx = \int_{\mathbb{R}^d} f_1\, dx + \int_{\mathbb{R}^d} f_2\, dx$.

To prove (a), we note that for $c \ge 0$, one has $(cf)^+ = cf^+$, $(cf)^- = cf^-$.

So if $c \ge 0$,

$$\int_{\mathbb{R}^d} cf\, dx = \int_{\mathbb{R}^d} (cf)^+ dx - \int_{\mathbb{R}^d} (cf)^- dx = \int_{\mathbb{R}^d} cf^+ dx - \int_{\mathbb{R}^d} cf^- dx$$
$$= c \int_{\mathbb{R}^d} f^+ dx - c \int_{\mathbb{R}^d} f^- dx = c \int_{\mathbb{R}^d} f\, dx.$$

On the other hand, if $c \le 0$, then $(cf)^+ = (-c) \cdot f^-$, $(cf)^- = (-c) f^+$. Since $-c \ge 0$, we have

$$\int_{\mathbb{R}^d} cf\, dx = \int_{\mathbb{R}^d} (-c) f^- dx - \int_{\mathbb{R}^d} (-c) f^+ dx$$
$$= (-c) \int_{\mathbb{R}^d} f^- dx - (-c) \int_{\mathbb{R}^d} f^+ dx$$
$$= c \int_{\mathbb{R}^d} f(x)\, dx.$$

Recall: For non-negative $f$, we proved $\int_{\mathbb{R}^d} cf\, dx = c \int_{\mathbb{R}^d} f\, dx$ for $c \ge 0$!

To prove (b), we denote $g(x) = f_1(x) + f_2(x)$. Then

$$g^+ - g^- = g = f_1 + f_2 = f_1^+ - f_1^- + f_2^+ - f_2^-$$
$$\Rightarrow g^+ + f_1^- + f_2^- = g^- + f_1^+ + f_2^+.$$
$$\Rightarrow \int_{\mathbb{R}^d} g^+ + \int_{\mathbb{R}^d} f_1^- + \int_{\mathbb{R}^d} f_2^- = \int_{\mathbb{R}^d} g^- + \int_{\mathbb{R}^d} f_1^+ + \int_{\mathbb{R}^d} f_2^+$$
$$\Rightarrow \int f_1 + f_2 = \int g = \int g^+ - \int g^- = \int f_1^+ - \int f_1^- + \int f_2^+ - \int f_2^- = \int f_1 + \int f_2.$$

Note: ① We used linearity in non-negative theory. ② We need all these integrals to be finite.

(2). Suppose $f_1 \le f_2$, then $f_2 - f_1 \ge 0$. So $\int_{\mathbb{R}^d} (f_2 - f_1)\, dx \ge 0$,

$$\Rightarrow \int_{\mathbb{R}^d} f_1\, dx \le \int_{\mathbb{R}^d} f_1\, dx + \int_{\mathbb{R}^d} (f_2 - f_1)\, dx = \int_{\mathbb{R}^d} f_2\, dx.$$

$f_1$ integrable $\Rightarrow -f_1$ integrable $\Rightarrow f_2 - f_1$ integrable.

(3). This follows from translation invariance in non-negative theory: ~~Denote $g(x) = f(x+y)$.~~

$$\int_{\mathbb{R}^d} f(x+y)\, dx = \int_{\mathbb{R}^d} f^+(x+y)\, dx - \int_{\mathbb{R}^d} f^-(x+y)\, dx$$
$$= \int_{\mathbb{R}^d} f^+(x)\, dx - \int_{\mathbb{R}^d} f^-(x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx. \quad \square$$

~~Rmk: According to (1), $L^1(\mathbb{R}^d)$ is a linear space.~~

· The following ~~property~~ "countable additivity w.r.t. domain" still holds.

**Prop.** Suppose $A_i \in \mathcal{L}$, $A_i \cap A_j = \phi$. Let $A = \bigcup_{i=1}^{\infty} A_i$.
If $f \in \mathcal{L}^1(A)$, then $\int_A f(x)dx = \sum_{i=1}^{\infty} \int_{A_i} f(x)dx$.

**Proof.** We first notice that $f \in L^1(A_i)$ since $\int_{\mathbb{R}^d} |f(x)\chi_{A_i}|\,dx \le \int_{\mathbb{R}^d} |f(x)\chi_A|\,dx < \infty$.
According to the "countable additivity w.r.t. domain" theorem in non-negative theory,

$$\sum_{i=1}^{\infty} \int_{A_i} f^{\pm}(x)\,dx = \int_A f^{\pm}(x)dx = \int_{\mathbb{R}^d} f^{\pm}(x)\chi_A(x)\,dx < +\infty.$$

It follows

$$\sum_{i=1}^{\infty} \int_{A_i} f(x)dx = \sum_{i=1}^{\infty} \left( \int_{A_i} f^+(x)dx - \int_{A_i} f^-(x)\,dx \right)$$

$$= \int_A f^+(x)dx - \int_A f^-(x)dx = \int_A f(x)\,dx. \qquad \square$$

· In non-negative theory, we have

$$\int_{\mathbb{R}^d} f(x)\,dx = 0 \iff f = 0 \text{ a.e.}$$

Obviously this is no longer true for general $f \in L^1(\mathbb{R}^d)$. However, we have

**Prop.** $f = 0$ a.e. on $A \iff \forall$ measurable subset $B \subset A$, $\int_B f\,dx = 0$.
[Assume $f$ is measurable on $A$].

**Proof.** If $f = 0$ a.e. on $A$, then $f^{\pm} = 0$ a.e. on $B$.

$$\Rightarrow \int_B f\,dx = \int_B f^+ dx - \int_B f^- dx = 0.$$

In fact, we proved: If $\int_B f \ge 0$ for any $B \subset A$, then $f \ge 0$ a.e.

If $\forall$ measurable subset $B \subset A$, we have $\int_B f\,dx = 0$.

$$\Rightarrow \int_{f>0} f\,dx = \int_A f^+ dx = 0, \quad \int_A f^- dx = \int_{f<0} (-f)dx = 0$$

$\Rightarrow f = 0$ a.e. in $\{x : f > 0\}$ and $f = 0$ a.e. in $\{x : f < 0\}$

$\Rightarrow f = 0$ a.e. $\square$

~~In the case $d=1$, one has even stronger result:~~
~~**Prop.** Suppose $f \in L^1([a,b])$, and $\int_{[a,c]} f(x)dx = 0$, $\forall c \in [a,b]$. Then $f = 0$ a.e. in $[a,b]$.~~
~~**Proof.**~~

In fact, we can prove a stronger result: It is enough to take the measurable subsets $B$ to be boxes.

**Prop.** Suppose $f \in L^1(\mathbb{R}^d)$, and $\int_B f(x)\,dx = 0$, $\forall$ box $B$. Then $f = 0$ a.e. in $\mathbb{R}^d$.

(**Rmk**: If $f$ is only defined on $A$, one can always perform zero extension of $f$)

**Proof:** We prove by contradiction.

> **Cor:** There is no measurable set $A \subset [0,1]$ s.t. $m(A \cap [a,b]) = \frac{b-a}{2}$, $\forall 0 \le a < b \le 1$
>
> **Proof:** If there exist such $A$, then $\int_{[a,b]} (\chi_A - \frac{1}{2}\chi_{[0,1]}) = 0$, $\forall [a,b]$
> $\Rightarrow \chi_A = \frac{1}{2}\chi_{[0,1]}$, contradiction. □

- WLOG, assume $m(\{x : f(x) > 0\}) = a > 0$.

  Since $\{x: f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : f(x) > \frac{1}{n}\}$, and $m(\{x: f(x)>0\}) = \lim_{n\to\infty} m(\{x: f(x) > \frac{1}{n}\})$,
  one can pick $n$ s.t. $m(\{x: f(x) > \frac{1}{n}\}) > \frac{a}{2}$.
- Now pick a closed set $F \subset \{x: f(x) > \frac{1}{n}\}$ s.t. $m(F) > \frac{a}{4}$. Then $\int_F f(x)dx \ge \frac{1}{n}\cdot\frac{a}{4} > 0$
- Let $U = \mathbb{R}^d \setminus F$. Then $U$ is open.

  By the structure theorem of open sets (c.f. Lecture 3), $U = \bigcup_{i=1}^{\infty} B_i$, where $B_i$'s are almost disjoint union of closed boxes.

  Let $U_1 = \bigcup_{i=1}^{\infty} \mathring{B}_i$. Then $U_1$ is the union of disjoint open sets, and $m(U \setminus U_1) = 0$.
- By condition,

$$
\begin{aligned}
0 = \int_{\mathbb{R}^d} f(x)\,dx &= \int_F f(x)dx + \int_U f(x)dx \\
&= \int_F f(x)\,dx + \int_{U\setminus U_1} f(x)\,dx + \int_{U_1} f(x)\,dx \\
&= \int_F f(x)dx + \underbrace{\int_{U\setminus U_1} f(x)dx}_{\substack{=0 \\ \text{since } m(U\setminus U_1)=0}} + \underbrace{\sum_{i=1}^{\infty}\int_{\mathring{B}_i} f(x)\,dx}_{\substack{=0 \\ \text{since } \mathring{B}_i \text{ is a box}}} \\
&= \int_F f(x)\,dx
\end{aligned}
$$

This is a contradiction. □

- Next we will prove the following "absolute continuity" property of Lebesgue's integral.

**Thm.** (Absolute continuity) Suppose $f \in L^1(A)$, where $A \subset \mathbb{R}^d$ is measurable. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for any measurable subset $B \subset A$ with $m(B) < \delta$, one has $\int_B |f|\,dx < \varepsilon$.

**Proof:**
- First assume $f \ge 0$.

  Then by non-negative theory, $\exists$ non-negative, simple function $\varphi(x) \le f(x)$ s.t.

  $$0 < \int_A (f - \varphi)\,dx = \int_A f(x)dx - \int_A \varphi(x)\,dx < \frac{\varepsilon}{2}.$$

  Since $\varphi$ is non-negative, simple and integrable, one can find $M$ s.t.

  $$\varphi(x) \le M \quad \text{a.e.}$$

  Take $\delta = \frac{\varepsilon}{2M}$. Then for any $B \subset A$, $B \in \mathcal{L}$ and $m(B) < \delta$, one has

  $$
  \begin{aligned}
  \int_B f\,dx &= \int_B (f-\varphi)dx + \int_B \varphi\,dx \le \int_A (f-\varphi)\,dx + \int_B \varphi dx \\
  &< \frac{\varepsilon}{2} + M\cdot\delta < \varepsilon.
  \end{aligned}
  $$
- For general $f$, one just apply the previous result to $f^+$, $f^-$, and use the fact $|f| = f^+ + f^-$. □

## 3. The Lebesgue Dominated Convergence Theorem (LDCT)

- Again we want to ask: If $f_n \to f$ a.e., can we say $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n\,dx = \int_{\mathbb{R}^d} f\,dx$ ?

At the end of last lecture, we studied the example

$$f_n(x) = \begin{cases} n, & 0<x<\frac{1}{n} \\ 0, & x\le 0 \text{ or } x\ge \frac{1}{n} \end{cases} \longrightarrow f(x)=0.$$

But $\int_{\mathbb{R}} f_n(x)\,dx = 1 \not\longrightarrow \int_{\mathbb{R}} f(x)\,dx = 0$

It turns out that the reason for the "non-convergence" is that the sequence of functions $\{f_n\}$ are NOT "controlled" by a single integrable function.

<u>Thm.</u> (Lebesgue Dominated Convergence Theorem)
Let $\{f_n\}$ be measurable functions on $\mathbb{R}^d$, and $f_n \to f$ a.e.
Suppose $\exists g \in L^1(\mathbb{R}^d)$ s.t. $|f_n(x)| \le g(x)$ a.e., $\forall n$.
Then $f \in L^1(\mathbb{R}^d)$, and $\lim_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,dx = \int_{\mathbb{R}^n} f(x)\,dx$.

NOTE: THIS IS THE THEOREM THAT WILL MAKE OUR LIVES MUCH EASIER

<u>Benefit 1</u>: You earn 10 minutes ☺