Real Analysis

Lecture 12
04/12/2018

Last time

- $\int_A f(x)\,dx = \int_A f^+(x)dx - \int_A f^-(x)\,dx$ (Assumption: Both are finite)
  ($\Leftrightarrow \int_A |f|\,dx < \infty$)

$\rightsquigarrow \|f\|_{L^1} = \int_A |f|\,dx$. "$L^1$-norm".

- Basic properties: ① $\int_A (c_1 f_1 + c_2 f_2)\,dx = c_1\int_A f_1\,dx + c_2\int_A f_2\,dx$, $\forall c_1, c_2 \in \mathbb{R}$.
  ② If $f_1 \le f_2$, then $\int_A f_1\,dx \le \int_A f_2\,dx$.
  ③ $\int_{\mathbb{R}^d} f(x+y)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$.
  ④ $f=0$ a.e. $\overset{\text{in } A}{\Leftrightarrow} \forall B \subset A$ measurable, $\int_B f\,dx = 0$
  $\Leftrightarrow \forall$ box B, $\int_B f\,dx = 0$.
- Countably additivity. $A_i \cap A_j = \phi$, $f \in L^1(\bigcup_{i=1}^\infty A_i)$ $\Longrightarrow \int_{\bigcup_{i=1}^\infty A_i} f\,dx = \sum_{i=1}^\infty \int_{A_i} f\,dx$. "signed measure".

Absolute continuity. $f \in L^1(A) \Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall m(B) < \delta$, $B \subset A$, one has $\int_B |f|\,dx < \varepsilon$.

Today: Convergence in "absolute integrability" theory

1. The Lebesgue Dominated Convergence Theorem

- Thm. (LDCT) Let $\{f_n\}$ be measurable functions on $\mathbb{R}^d$, and $f_n \to f$ a.e. Suppose $\exists g \in L^1(\mathbb{R}^d)$ s.t. $|f_n(x)| \le g(x)$ a.e., $\forall n$. Then $f \in L^1(\mathbb{R}^d)$, and $\lim_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)dx = \int_{\mathbb{R}^d} f(x)\,dx$.

Proof: - Let $A = \{x : \lim_{n\to\infty} f_n(x) = f(x)\}$. Then $m(A^c) = 0$.
Since $|f|\cdot\chi_A = \lim_{n\to\infty} |f_n|\cdot\chi_A \le g\,\chi_A$, the monotonicity implies

$$\int_{\mathbb{R}^d}|f| = \int_A |f| = \int_{\mathbb{R}^d}|f|\chi_A \le \int_{\mathbb{R}^d} g\chi_A \le \int_{\mathbb{R}^d} g < +\infty$$

So $f \in L^1(\mathbb{R}^d)$.

- Now we apply Fatou's lemma to sequence $\{g - f_n\}$ and $\{g + f_n\}$:
[Note: $g \pm f_n \ge 0$, $\forall n$, on the set $\mathbb{R}^d - \underbrace{\bigcup_{n=1}^\infty \{x : |f_n(x)| > g(x)\}}_{\text{measure zero}} =: A_1$, with $m(A_1^c) = 0$.
So we work on A_1 below.]

$$\int_{A_1} g + \int_{A_1} f = \int_{A_1}\liminf_{n\to\infty}(g+f_n) \le \liminf_{n\to\infty}\int_{A_1}(g+f_n) = \int_{A_1} g + \liminf_{n\to\infty}\int_{A_1} f_n$$

$$\int_{A_1} g - \int_{A_1} f = \int_{A_1}\liminf_{n\to\infty}(g-f_n) \le \liminf_{n\to\infty}\int_{A_1}(g-f_n) = \int_{A_1} g - \limsup_{n\to\infty}\int_{A_1} f_n$$

$$\Rightarrow \limsup_{n\to\infty}\int_{\mathbb{R}^d} f_n = \limsup_{n\to\infty}\int_{A_1} f_n \le \int_{A_1} f \le \liminf_{n\to\infty}\int_{A_1} f_n = \liminf_{n\to\infty}\int_{\mathbb{R}^d} f_n. \quad \square$$

Cor. Under the same assumption of LDCT, we have a stronger conclusion
$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n - f|\, dx = 0.$$

Proof. Since $|f_n - f| \to 0$ a.e., and $|f_n - f| \le 2g(x)$ a.e., we can apply LDCT to $\{|f_n - f|\}$ to conclude
$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n - f|\, dx = 0. \quad \square$$

· **Def.** We say $f_n \to f$ <u>in L^1-norm</u> (or f_n converges to f <u>in the mean</u>) if
$$\|f_n - f\|_{L^1} = \int_{\mathbb{R}^d} |f_n - f|\, dx \to 0.$$

Prop. Let f_n be a sequence of integrable functions. If $f_n \to f$ in L^1-norm, then $f_n \to f$ in measure.

$\boxed{\text{Lec. 7}}$ $\forall \varepsilon > 0$, $m(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0$.

Proof. According to the Markov's inequality (Lec. 9), we have
$$m(\{x : |f_n(x) - f(x)| \ge \lambda\}) \le \frac{1}{\lambda} \int_{\mathbb{R}^d} |f_n - f|\, dx.$$
So $f_n \to f$ in L^1-norm $\Rightarrow$ $f_n \to f$ in measure. $\square$

Cor. If $f_n \to f$ in L^1-norm, then $\exists$ subsequence $f_{n_k} \to f$ a.e.

Rmk. Uniform convergence $\not\Rightarrow$ convergence in L^1-norm.

[So: none of the convergence modes we studied in Lec. 7 implies L^1-convergence]

Example: $f_n = \frac{1}{n} \chi_{[n, 2n]} \to 0$ uniformly.
But $\int_{\mathbb{R}} |f_n|\, dx = 1 \not\to 0$.

"converse" to each other

· We can also prove the following version of DCT. (with "a.e. convergence" replaced by "convergence in measure")

Thm. (DCT, "convergence in measure" version.)
Let $\{f_n\}$ be measurable functions on $\mathbb{R}^d$, s.t. $f_n \to f$ in measure.
Suppose $\exists\, g \in L^1(\mathbb{R}^d)$ s.t. $|f_n(x)| \le g(x)$, a.e., $\forall n$.
Then $f \in L^1(\mathbb{R}^d)$, and $f_n \to f$ in L^1-norm.

Please read the 2nd proof given in the book.

Proof. Since $f_n \to f$ in measure, one can find a subsequence f_{n_k} s.t.
$$f_{n_k} \to f \text{ a.e.}$$
It follows from LDCT that $f \in L^1(\mathbb{R}^d)$, and $\|f_{n_k} - f\|_{L^1} \to 0$.
~~Now we have $|f_{n_k} - f| \le 2g$ a.e.~~

Argue by contradiction.

· Note that the above argement holds for any subsequence of f_n. In other words, we showed that ANY subsequence of f_n has a subsequence that converges in L^1-norm to f.
So we must have $f_n \to f$ in L^1-norm. $\square$

Rmk: There is a 2nd proof, which is a direct proof, i.e. without using LDCT.

The idea is the following:

· We want to prove $\int |f_n - f|\, dx < \varepsilon$ for n large.

· Usually, to prove that such an integral is small, one can decompose the integral into 3 pieces, one piece is small by itself (……), the second piece is the integral of a small quantity over a bounded set and thus is small, the third piece is the integral of a bounded quantity over a small set and thus is also small.

Back to this problem, one has

$$\int_{\mathbb{R}^d} |f_n - f| \le \underbrace{\int_{\{x : |f_n - f| \ge \varepsilon\}}}_{\substack{\text{small set} \\ \text{since } f_n \to f \text{ in measure}}} \underbrace{|f_n - f|}_{\text{bounded by } 2g} + \underbrace{\int_{\{x : |f_n - f| < \varepsilon\}}}_{\boxed{\text{Maybe unbounded! Need more work}}} \underbrace{|f_n - f|}_{\text{small}}.$$

To bound the second term, one use the fact that

$$f_n - f \in L^1 \Rightarrow \exists N \text{ s.t. } \int_{|x| > N} |f_n - f| < \varepsilon.$$

So one can write

$$\int_{\{x : |f_n - f| < \varepsilon\}} |f_n - f| \le \underbrace{\int_{|x| > N} |f_n - f|}_{\text{"small by itself"}} + \underbrace{\int_{\{x : |x| < N,\ |f_n - f| < \varepsilon\}}}_{\text{bounded now!}} \underbrace{|f_n - f|}_{\text{small}}$$

This is the idea!

For details, please read P157-158 of your book.

2. Completeness of $L^1(A)$.

- <u>Recall</u>: • A sequence $\{x_n\}$ of numbers is a <u>Cauchy sequence</u> if $\forall \varepsilon>0$, $\exists N$ s.t. $\forall n, m > N$, one has $|x_n - x_m| < \varepsilon$.
  - The <u>completeness</u> of $\mathbb{R}$: Any Cauchy sequence in $\mathbb{R}$ is a convergent sequence.
- Now we extend these conceptions to $L^1(A)$.

<u>Def</u>: A sequence of functions $f_n \in L^1(A)$ is a <u>Cauchy sequence</u> if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n, m > N$, one has $\|f_n - f_m\|_{L^1} < \varepsilon$.

<u>Thm</u>. (Completeness of $L^1(A)$) (Riesz-Fischer Thm)
Let $\{f_n\}$ be a Cauchy sequence in $L^1(A)$. Then $\exists f \in L^1(A)$ s.t.
$$\lim_{n\to\infty} \|f_n - f\|_{L^1(A)} = 0.$$

<u>Proof</u>: • ~~Pick a subsequence f_{n_k}~~ Since $\{f_n\}$ is a Cauchy sequence, for any k, $\exists N(k)$ s.t. $\forall n, m > N(k)$, one has $\|f_n - f_m\|_{L^1(A)} < \frac{1}{2^{k+1}}$.
We can always arrange $N(1) < N(2) < N(3) < \cdots$.
Now for any k, pick $n_k > N(k)$. Then we get a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, such that $\|f_{n_k} - f_{n_{k-1}}\|_{L^1(A)} = \int_A |f_{n_k}(x) - f_{n_{k-1}}(x)|\,dx < \frac{1}{2^k}$.
Now we let
$$f(x) = f_{n_1}(x) + \sum_{k=2}^{\infty} \left(f_{n_k}(x) - f_{n_{k-1}}(x)\right)$$
$$g(x) = |f_{n_1}(x)| + \sum_{k=2}^{\infty} \left|f_{n_k}(x) - f_{n_{k-1}}(x)\right|.$$
Since $\int_A |f_{n_1}(x)|dx + \sum_{k=2}^{\infty} \int_A |f_{n_k}(x) - f_{n_{k-1}}(x)|\,dx \le \int_A |f_{n_1}(x)|\,dx + 1 < \infty$, the monotone convergence thm implies that $g \in L^1(A)$.
[So g is finite a.e., which implies that the series defining f converges a.e.]
Since $|f| \le g$, we see $f \in L^1(A)$.
Since $f_{n_k} = f_{n_1}(x) + \sum_{i=2}^{k} (f_{n_i}(x) - f_{n_{i-1}}(x)) \to f$ a.e., we apply LDCT to conclude $\|f_{n_k} - f\|_{L^1} \to 0$ as $k \to \infty$.
use: $|f - f_{n_k}| \le g$. So g is a dominated.

- In particular, $\exists K$ s.t. $\forall n > K$, $\|f_{n_k} - f\|_{L^1} < \varepsilon/2$.
  Since $\{f_n\}$ is Cauchy, $\exists N_\varepsilon$ s.t. $\forall n, m > N_\varepsilon$, $\|f_n - f_m\|_{L^1} < \varepsilon/2$.
  $\Rightarrow \forall n > \max\{n_K, N_\varepsilon\}$, $\|f_n - f\|_{L^1} < \varepsilon$. $\square$