Real Analysis

Last time.

Midterm:
04/28.
19:00-21:30
@ 5402, 5401

- **Lebesgue Dominated Convergence Theorem.**

$$\left.\begin{array}{l} f_n \to f \text{ a.e. [or in measure]} \\ \exists g \in L^1 \text{ s.t. } |f_n| \le g \text{ a.e.} \end{array}\right\} \Rightarrow f \in L^1, \text{ and } \|f_n - f\|_{L^1} \to 0$$

$\hookrightarrow$ • $\int f_n dx \to \int f dx$.

• $f_n \to f$ in measure.

- **Completeness of L^1 (and L^p, $p \ge 1$) [Riesz-Fischer]**

$\{f_n\} \subset L^1$ (or L^p) is Cauchy $\Rightarrow \exists f \in L^1$ (or L^p) s.t. $f_n \to f$ in L^1-norm (or L^p-norm).

Today: Applications of LDCT.

1. Countable additivity

- **Prop.** $f \in L^1(A)$, $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \cap A_j = \phi \Rightarrow \int_A f\,dx = \sum_{i=1}^{\infty} \int_{A_i} f\,dx$.

2nd Proof: Let $\tilde{f}_i = f \cdot \chi_{A_i}$ and $f_n = \sum_{i=1}^{n} \tilde{f}_i$. Then $|f_n| \le f$, and $f_n \to f$.

So by LDCT,

$$\int_A f\,dx = \lim_{n\to\infty} \int_A f_n\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} f\,dx = \sum_{i=1}^{\infty} \int_{A_i} f\,dx. \quad \square$$

- **Prop.** Suppose $f_n \in L^1(A)$ and $\sum_{n=1}^{\infty} \int_A |f_n(x)|\,dx < +\infty$. Then $\exists f \in L^1(A)$ s.t. $\sum_{n=1}^{\infty} f_n$ converges a.e. to f, and $\sum_{n=1}^{\infty} \int_A f_n(x)\,dx = \int_A f(x)\,dx$.

Proof. • Let $F = \sum_{n=1}^{\infty} |f_n|$. Then by non-negative theory (Lec. 10),

$$\int_A F\,dx = \sum_{n=1}^{\infty} \int_A |f_n|\,dx < +\infty$$

$\Rightarrow F \in L^1(A) \Rightarrow F$ is a.e. finite.

In other words, for a.e. x, $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely.

So $\exists$ function f s.t. $\sum_{n=1}^{\infty} f_n$ converges to f a.e.

• Since $f = \lim_{N\to\infty} \sum_{n=1}^{N} f_n$, and each $\sum_{n=1}^{N} f_N$ is measurable $\Rightarrow f$ is measurable.

Since $|f| \le \sum_{n=1}^{\infty} |f_n| = F \in L^1(A)$, we see $f \in L^1(A)$.

• Finally let $g_n = \sum_{i=1}^{n} f_i$. Then $|g_n| \le \sum_{i=1}^{n} |f_i| \le F$, and $g_n \to f$ a.e.

So by LDCT,

$$\int_A f\,dx = \lim_{n\to\infty} \int_A g_n\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \int_A f_i\,dx = \sum_{n=1}^{\infty} \int_A f_n\,dx. \quad \square$$

2. Riemann integrals v.s. Lebesgue integrals

- Let $f: [a,b] \to \mathbb{R}$ be a bounded function.

Recall: • For each partition $P: a = x_0 < x_1 < \cdots < x_n = b$, the upper/lower Darboux sums are

$$\overline{S}_P(f) = \sum_{k=1}^{n} \sup_{x_{k-1} \le x \le x_k} f(x) \cdot (x_k - x_{k-1}), \qquad \underline{S}_P(f) = \sum_{k=1}^{n} \inf_{x_{k-1} \le x \le x_k} f(x) \cdot (x_k - x_{k-1})$$

$\rightsquigarrow$ the upper/lower Darboux integrals

$$\overline{\int_a^b} f\,dx = \inf_P \overline{S}_P(f), \qquad \underline{\int_a^b} f\,dx = \sup_P \underline{S}_P(f).$$

• f is Riemann integrable $\Leftrightarrow$ $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx \quad \left(=: \int_a^b f\,dx\right)$

- Now given a partition P, we can define two step functions

$$\overline{\Phi}_P(f) = \sum_{k=1}^{n} \sup_{x_{k-1} \le x \le x_k} f(x) \cdot \chi_{[x_{k-1}, x_k)}, \qquad \underline{\Phi}_P(f) = \sum_{k=1}^{n} \inf_{x_{k-1} \le x \le x_k} f(x) \cdot \chi_{[x_{k-1}, x_k)}$$

Then by definition, one has

$$\int_{[a,b]} \overline{\Phi}_P(f)\,dx = \overline{S}_P(f), \qquad \int_{[a,b]} \underline{\Phi}_P(f)\,dx = \underline{S}_P(f)$$

- By the trick of "common refinement" [Given partitions P and P', one can make a new partition whose nodes are the union of nodes of P and P'.]

one can pick a sequence of partitions with $P_1 \le P_2 \le \cdots$ (meaning that each P_{n+1} is a refinement of P_n) s.t.

$$\overline{\int_a^b} f\,dx = \lim_{k\to\infty} \overline{S}_{P_k}(f) = \lim_{k\to\infty} \int_{[a,b]} \overline{\Phi}_{P_k}(f)\,dx, \qquad \underline{\int_a^b} f\,dx = \lim_{k\to\infty} \underline{S}_{P_k}(f) = \lim_{k\to\infty} \int_{[a,b]} \underline{\Phi}_{P_k}(f)\,dx.$$

One can check • $\overline{\Phi}_{P_k}(f) \searrow \overline{\Phi}(f)$, $\underline{\Phi}_{P_k}(f) \nearrow \underline{\Phi}(f)$ for some measurable functions $\overline{\Phi}(f)$ and $\underline{\Phi}(f)$, with $\boxed{\underline{\Phi}(f) \le f \le \overline{\Phi}(f)}$

• f is Riemann integrable $\Leftrightarrow$ $0 = \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx = \lim_{k\to\infty} \int_{[a,b]} \left(\overline{\Phi}_{P_k}(f) - \underline{\Phi}_{P_k}(f)\right) dx$

used the fact $m([a,b]) < \infty$ and monotone convergence (or LDCT)

$$= \int_{[a,b]} \lim_{k\to\infty} \left(\overline{\Phi}_{P_k}(f) - \underline{\Phi}_{P_k}(f)\right) dx = \int_{[a,b]} \left(\overline{\Phi}(f) - \underline{\Phi}(f)\right) dx$$

$\Leftrightarrow$ $\overline{\Phi}(f) = \underline{\Phi}(f)$ a.e. ($\Leftrightarrow f$ is a.e. continuous.)

used the fact $\overline{\Phi}(f) \ge \underline{\Phi}(f)$

As a consequence, we get

Prop. If a bounded function f on $[a,b]$ is Riemann integrable, then it is Lebesgue integrable on $[a,b]$, and $\int_{[a,b]} f\,dx = \int_a^b f(x)\,dx$.

Proof: f is Riemann integral $\Rightarrow$ $\overline{\Phi}(f) = \underline{\Phi}(f) = f$ a.e. $x \in [a,b]$.

By LDCT, $\int_{[a,b]} \underline{\Phi}_{P_k}(f)\,dx \to \int_{[a,b]} \underline{\Phi}(f)\,dx = \int_{[a,b]} f\,dx.$ $\square$

Rmk: There exists functions f so that the improper integral $\int_a^b f\,dx$ (in Riemann sense) exists, while f is NOT Lebesgue integrable. For example, take

$$f: (0, +\infty) \to \mathbb{R}, \qquad f(x) = \frac{(-1)^n}{n} \chi_{[n-1, n)}.$$

$$\Rightarrow \int_0^\infty f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2, \qquad \text{but } \int_{[0,\infty)} |f|\,dx = +\infty.$$

3. Integrals with parameter

- Very often, one need to study integrals of the form
$$F(t) = \int_A f(t,x)\,dx,$$
where t is viewed as a parameter.

<u>Prop.</u> Let $f: A \times B \longrightarrow \mathbb{R}$ be a function, $\overbrace{\text{such that}}^{\text{where } A, B \in \mathcal{L},}$

① $\forall t \in A$: $f_t(x) := f(t,x)$ is measurable

② $\forall x \in B$: $f_{(x)}(t) := f(t,x)$ is continuous

Suppose $\exists g \in L^1(B)$ s.t. $|f(t,x)| \le g(x)$, $\forall t \in A$, $x \in B$.
Then for $\forall t \in A$, $f_t \in L^1(B)$, and the function
$$F(t) := \int_B f(t,x)\,dx$$
is continuous w.r.t. $t \in A$.

<u>Proof.</u> Suppose $t_n \in A$ and $t_n \to t_0 \in A$. Then
$$f(t_n, x) = f_{t_n}(x) \longrightarrow f_{t_0}(x) = f(t_0, x) \quad \underline{\text{AND}} \quad |f(t_n,x)| \le g(x) \in L^1.$$
By LDCT, we have $F(t_n) \to F(t_0)$. $\square$

<u>Prop.</u> Let $f: (a,b) \times A \longrightarrow \mathbb{R}$ be a function s.t.

① $\forall t \in (a,b)$: $f_t(x) := f(t,x)$ is integrable.

② $\forall x \in A$: $f_{(x)}(t) := f(t,x)$ is differentiable

Suppose $\exists g \in L^1(A)$ s.t. $\left|\frac{d}{dt} f(t,x)\right| \le g(x)$, $\forall t \in (a,b)$, $x \in A$.
Then for $\forall t \in (a,b)$,
$$\frac{d}{dt}\int_A f(t,x)\,dx = \int_A \frac{d}{dt} f(t,x)\,dx.$$

<u>Proof.</u> For any $(t,x) \in (a,b) \times A$, one has
$$\frac{d}{dt} f(t,x) = \lim_{h_k \to 0} \frac{f(t+h_k, x) - f(t,x)}{h_k}.$$
By mean value theorem,
$$\left|\frac{f(t+h_k, x) - f(t,x)}{h_k}\right| \le g(x), \quad \forall x \in A.$$
So by LDCT,
$$\frac{d}{dt}\int_A f(t,x)\,dx = \lim_{k \to \infty} \int_A \frac{f(t+h_k, x) - f(t,x)}{h_k}\,dx = \int_A \frac{d}{dt} f(t,x)\,dx. \ \square$$

4. The theorems of ~~Tonelli and~~ Fubini

- Now let $f=f(x,y)$ be a "2-variable" integrable function defined on $A\times B\subset\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$. A natural question is: Can we evaluate the integral $\int_{A\times B} f(x,y)\,dxdy$ via iterated integral $\int_A\left(\int_B f(x,y)\,dy\right)dx$?

  Note: We need 3 results: ① $f_x(y):=f(x,y)$ is integrable w.r.t. $y\in B$

  ② $\int_B f(x,y)\,dy$ is integrable w.r.t. $x\in A$

  ③ $\int_{A\times B} f(x,y)\,dxdy=\int_A\left(\int_B f(x,y)\,dy\right)dx$.

  Example: Consider $f(x,y)=\dfrac{x^2-y^2}{(x^2+y^2)^2}$, $(x,y)\in[0,1]\times[0,1]$.

  By using formulas
$$\int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}+C$$
$$\int\frac{dx}{(x^2+a^2)^2}=\frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^2}\int\frac{dx}{x^2+a^2}$$
  one can show
$$\int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy=\int_0^1\left(-\frac{1}{1+y^2}\right)dy=-\frac{\pi}{4}$$
$$\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx=\int_0^1\frac{1}{1+x^2}\,dx=\frac{\pi}{4}$$
$\neq$

  $\boxed{\text{Reason}}$
$$\int_0^1\left(\int_0^1|f(x,y)|\,dx\right)dy=+\infty.\quad(\longrightarrow\text{Not integrable}).$$

- The main theorem for multiple integrals Lebesgue is that as long as the integrand is integrals, you can always evaluate the integral via iterated integrals, and you can exchange the order of integration in whatever way you want.

  Thm. (Fubini) Let $A\in\mathbb{R}^{d_1}$, $B\subset\mathbb{R}^{d_2}$ be measurable, and $f=f(x,y)\in L^1(A\times B)$.

  Then (1) For a.e. $y\in B$, $f_y(x):=f(x,y)\rightsquigarrow f_y\in L^1(A)$.

  (2) As a function of y, $F(y):=\int_A f(x,y)\,dx\longrightarrow F\in L^1(B)$.

  (3) We have
$$\int_{A\times B} f(x,y)\,dxdy=\int_B\left(\int_A f(x,y)\,dx\right)dy=\int_B\left(\int_A f(x,y)\,dx\right)dy.$$

  Rmk: · By performing zero-extension, it is enough to prove the theorem for $A=\mathbb{R}^{d_1}$, $B=\mathbb{R}^{d_2}$.

  · Idea of proof: from special to general.

  · By symmetry, for (3) it is enough to check $\int_{A\times B} f\,dxdy=\int_B\left(\int_A f(x,y)\,dx\right)dy$.

Proof: WLOG, we assume $A = \mathbb{R}^{d_1}$, $B = \mathbb{R}^{d_2}$.

For simplicity we let $\mathcal{H} = \{ f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) : f \text{ satisfies (1), (2), (3)} \}$

(path of proof: $\chi_{B_1 \times B_2} \in \mathcal{H} \to \chi_{G_\delta} \in \mathcal{H} \to \chi_{\text{measurable}} \in \mathcal{H} \to \text{simple} \in \mathcal{H} \to \text{integrable} \in \mathcal{H}$;
$\chi_{B_1 \times B_2} \in \mathcal{H} \to \chi_{\text{open}} \in \mathcal{H} \to \chi_{\text{measure } 0} \in \mathcal{H}$)

Step 1. Let $f = \chi_{B_1 \times B_2}$, where $B_1 \subset \mathbb{R}^{d_1}$, $B_2 \subset \mathbb{R}^{d_2}$ are (bounded) boxes. $\to f \in \mathcal{H}$.

Then (1) $\forall y \in \mathbb{R}^{d_2}$, $f_y(x) = \chi_{B_1}(x) \Rightarrow f_y \in L^1(\mathbb{R}^{d_1})$ (since B_1 is bounded)

(2) $F(y) = \int_{\mathbb{R}^{d_1}} f(x,y)\,dx = m(B_1) \cdot \chi_{B_2} \Rightarrow F \in L^1(\mathbb{R}^{d_2})$ (since B_2 is bounded)

(3)
$$\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x,y)\,dx\,dy = m(B_1 \times B_2) = m(B_1) \times m(B_2)$$

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^{d_2}} m(B_1)\, \chi_{B_2}(y)\,dy = m(B_1) \times m(B_2)$$

Step 2. If $f_1, f_2 \in \mathcal{H}$, then $c_1 f_1 + c_2 f_2 \in \mathcal{H}$. $\longrightarrow f_1, \cdots, f_n \in \mathcal{H} \Rightarrow \sum_{i=1}^n c_i f_i \in \mathcal{H}$.

This follows from the linearity of Lebesgue integrals.

Step 3. Suppose either $f_n \in \mathcal{H}$, $f_n \nearrow f$, $f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$

or $f_n \in \mathcal{H}$, $f_n \searrow f$, $f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, then $f \in \mathcal{H}$.

In fact: If $f_n \nearrow f$, then $f_n - f_1 \nearrow f - f_1$, and $f_n - f_1 \geq 0$.

So by monotone convergence theorem, [use the fact $\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f_1\,dx\,dy < +\infty$]

$$\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f_n(x,y)\,dx\,dy \longrightarrow \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x,y)\,dx\,dy \quad (\star)$$

(1a) For a.e. $y \in \mathbb{R}^{d_2}$, $(f_n)_y(x) = f_n(x,y)$, then $(f_n)_y \in L^1(\mathbb{R}^{d_1})$, and $(f_n)_y$ converges to a measurable function f_y, with $f_y - f_1 \geq 0$.

Moreover, the monotone convergence theorem implies

$$F_n(y) := \int_{\mathbb{R}^{d_1}} f_n(x,y)\,dx \longrightarrow \int_{\mathbb{R}^{d_1}} f_y(x)\,dx := F(y)$$

(2) Since each $F_n \in L^1(\mathbb{R}^{d_2})$, and $F_1 \leq F_2 \leq \cdots$, the monotone convergence thm implies

$$\int_{\mathbb{R}^{d_2}} F_n(y)\,dy \longrightarrow \int_{\mathbb{R}^{d_2}} F(y)\,dy \quad (\star\star).$$

It follows that $\int_{\mathbb{R}^{d_2}} F(y)\,dy = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x,y)\,dx\,dy < +\infty$

So $F \in L^1(\mathbb{R}^{d_2})$.

(1b) So for a.e. y, $F(y)$ is finite. $\Rightarrow$ for a.e. y, $f_y \in L^1(\mathbb{R}^{d_1})$.

(3)
$$\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x,y)\,dx\,dy = \int_{\mathbb{R}^{d_2}} F(y)\,dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right) dy$$

To prove the case $f_n \searrow f$, one only need to notice $-f_n \nearrow -f$.

$\boxed{\text{Step 4}}$ If E is a subset in the boundary of a box (so in particular $m(E)=0$), then $\chi_E \in \mathcal{F}$.
This can be verified from definition.

$\boxed{\text{Step 5}}$ If U is a bounded open set, then $\chi_U \in \mathcal{F}$. Note: U is bounded open $\Rightarrow \chi_U \in L^1$

To see this, we write $U = \bigcup_{k=1}^{\infty} \overline{B}_k = \bigcup_{k=1}^{\infty} B_k$, where $\overline{B}_k$'s are almost disjoint closed boxes, B_k's are disjoint boxes (by removing some overlaping body) and $\mathring{\overline{B}}_k \subset B_k \subset \overline{B}_k$.

By Step 1, Step 2, Step 4, $\chi_{B_k} \in \mathcal{F}$.

By Step 2, $\chi_{\bigcup_{k=1}^{\infty} B_k} \in \mathcal{F}$.

By Step 3, $\chi_U \in \mathcal{F}$.

$\boxed{\text{Step 6}}$ For any bounded measure zero set E, $\chi_E \in \mathcal{F}$.

We first choose bounded open sets $U_1 \supset U_2 \supset \cdots$ s.t. $E \subset \bigcap_{n=1}^{\infty} U_n$, and $m(\bigcap_{n=1}^{\infty} U_n) = 0$.

By Step 3 and Step 5, $\bigcap_{n=1}^{\infty} U_n \in \mathcal{F} \Rightarrow \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_{\bigcap_{n=1}^{\infty} U_n}(x,y)\, dx \right) dy = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \chi_{\bigcap_{n=1}^{\infty} U_n}\, dx dy = 0$.

$\Rightarrow \int_{\mathbb{R}^{d_1}} \chi_{\bigcap_{n=1}^{\infty} U_n}(x,y)\, dx = 0$ for a.e. y.

$\Rightarrow \int_{\mathbb{R}^{d_1}} \chi_E(x,y)\, dx = 0$ for a.e. y $\Rightarrow$ χ_E satisfies (1), (2), and

$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E(x,y)\, dx \right) dy = \int_{\mathbb{R}^{d_2}} 0\, dx = 0 = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \chi_E(x,y)\, dx dy$. So $\chi_E \in \mathcal{F}$.

$\boxed{\text{Step 7}}$ For any measure 0 set E, $\chi_E \in \mathcal{F}$.

This follows from Step 3 and the fact $E = \bigcup_{n=1}^{\infty} (B(0,n) \cap E)$.

$\boxed{\text{Step 8}}$ For any measurable set A with $m(A) < +\infty$, $\chi_A \in \mathcal{F}$.

In fact, we can always write $A = G \setminus E$, where $G = \bigcap_{n=1}^{\infty} U_n$ is a G_δ set, and $m(E)=0$.

Let $G_K = G \cap B(0,K)$. Then $A = (\bigcup_K G_K) \setminus E$.

By Step 5, Step 3, Step 7, Step 2 and Step 3 again, we see $\chi_A \in \mathcal{F}$.

$\boxed{\text{Step 9}}$ Any integrable simple function $f \in \mathcal{F}$.

This is a consequence of Step 2 and Step 8.

$\boxed{\text{Step 10}}$ Any integrable function $f \in \mathcal{F}$.

To see this, we can write $f = f^+ - f^-$, where f^+, f^- are non-negative and integrable.

By Step 9, Step 3, $f^+, f^- \in \mathcal{F}$.

By Step 2, $f \in \mathcal{F}$. $\square$