Real Analysis

Last time

- Countable additivity:
  - $f \in L^1(\bigcup_{n=1}^{\infty} A_n) \Rightarrow \int_{\bigcup_{n=1}^{\infty} A_n} f\,dx = \sum_{n=1}^{\infty} \int_{A_n} f\,dx$
  - $\sum_{n=1}^{\infty} \int_A |f_n|\,dx < \infty \Rightarrow \int_A \sum_{n=1}^{\infty} f_n\,dx = \sum_{n=1}^{\infty} \int_A f_n\,dx$.
- Riemann integral: A bounded function f on $[a,b]$ is Riemann integrable

  $\Leftrightarrow$ $m(\{x : f \text{ is discontinuous at } x\}) = 0$

  $\Rightarrow$ $f \in L^1([a,b])$, and $\int_{[a,b]} f\,dx = \int_a^b f(x)\,dx$.
- Integral with parameter: Under very weak assumptions (natural assumption in t. ~~nature~~ weak assumption in x, dominate by $g \in L^1(A)$),

  $$F(t,x) = \int_A f(t,x)\,dx$$

  is continuous ~~xx~~/differentiable in t.
- Fubini's Theorem: Suppose $f \in L^1(A \times B)$. Then
  (1) For a.e. $y \in B$, $f(\cdot, y) \in L^1(A)$
  (2) The function $F(y) = \int_A f(x,y)\,dx \rightsquigarrow F \in L^1(B)$
  (3) $\int_{A\times B} f(x,y)\,dxdy = \int_B \left(\int_A f(x,y)\,dx\right) dy = \int_A \left(\int_B f(x,y)\,dy\right) dx$.

Today: More on Lebesgue integral

1. Consequences of Fubini's Theorem

- We have the following non-negative version of Fubini's theorem.

Thm (Tonelli) Suppose $f = f(x,y)$ is non-negative and measurable on $A \times B$. Then
(1) For a.e. $y \in B$, $f(\cdot, y)$ is measurable on A.
(2) The function $F(y) = \int_A f(x,y)\,dx$ is measurable on B.
(3) $\int_{A\times B} f(x,y)\,dxdy = \int_B \left(\int_A f(x,y)\,dx\right) dy = \int_A \left(\int_B f(x,y)\,dy\right) dx$ [May be $+\infty$!]

Proof: [Idea: · To apply Fubini's theorem, we need to approximate f by integrable functions.
· We can try to approximate by an increasing sequence so that we can use MCT.
· Given a non-negative function, how to get an integrable one? cut both the domain and the range of the function!]

- Consider the truncated functions

$$f_k(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in \cancel{B(0,k)}\ B(0,k)\cap(A\times B) \text{ AND } f(x,y) < k \\ k, & \text{if } (x,y) \in B(0,k)\cap(A\times B), \text{ and } f(x,y) \geq k \\ 0, & \cancel{\text{otherwise}}\ \text{otherwise} \end{cases}$$

Then each $f_k \in L^1(A\times B)$, and $f_1 \leq f_2 \leq f_3 \leq \cdots \nearrow f$.

- By Fubini (1), $\exists B_k \subset B$ with $m(B_k)=0$, s.t. $f_k(\cdot, y) \in L^1(A)$ for $y \in B \setminus B_k$.
It follows that for $y \in B \setminus \bigcup_{k=1}^{\infty} B_k$, i.e. for a.e. y, (since $m(\bigcup_{k=1}^{\infty} B_k)=0$), the function $f(\cdot, y) = \lim_{k\to\infty} f_k(\cdot, y)$ is measurable.
- Since $f_k(\cdot, y) \nearrow f(\cdot, y)$, the MCT gives
$$F_k(y) := \int_A f_k(x,y)\,dx \nearrow \int_A f(x,y)\,dy =: F(y)$$
By Fubini (2), the function F_k is measurable. So F is measurable on B.
- Applying MCT again, we get
$$\int_B \left(\int_A f_k(x,y)\,dx\right) dy \to \int_B\left(\int_A f(x,y)\,dx\right)dy$$
By Fubini (3), we know
$$\int_B \left(\int_A f_k(x,y)\,dx\right) dy = \int_{A\times B} f_k(x,y)\,dx\,dy \xrightarrow{\text{MCT again!}} \int_{A\times B} f(x,y)\,dx\,dy.$$
So we must have
$$\int_{A\times B} f(x,y)\,dx\,dy = \int_B\left(\int_A f(x,y)\,dx\right)dy.$$
$\square$

Rmk. In applications, to evaluate a multiple integral, one may combine Tonelli and Fubini:
- First apply Tonelli to $|f|$, so that one can compute the multiple integral of $|f|$ via iterated integrals. Try to show that the result is NOT infinite
- As a consequence, f is integral. Then apply Fubini to the multiple integral of f, i.e. evaluate the integral of f via iterated integrals.

- Cor. If $A \subset \mathbb{R}^{d_1}$, $B \subset \mathbb{R}^{d_2}$ are measurable, then $A\times B \subset \mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ is measurable, and
$$m(A\times B) = m(A)\times m(B)$$

Proof. • We can write $A = \bigcup_{n=0}^{\infty} A_n$, where $m(A_0)=0$, and each A_n $(n\geq 1)$ is closed in $\mathbb{R}^{d_1}$
$B = \bigcup_{n=0}^{\infty} B_n$, --- $m(B_0)=0$, --- B_n ------- in $\mathbb{R}^{d_2}$.
(Note: By replacing A_n or B_n with $(\bigcup_{i=1}^{n} A_i) \cap B(0,n)$, or $(\bigcup_{i=1}^{n} B_i) \cap B(0,n)$, we may assume that each A_n / B_n is bounded and closed.)
- Then each $A_n \times B_m$ $(n,m\geq 1)$ is closed, and thus measurable.
  - each $A_0\times B_n$ is measurable with measure 0, since $\exists$ boxes C_i, $D_j^{(n)}$ s.t. $A_0 \subset \bigcup_i C_i$, $B_n \subset \bigcup_j D_j^{(n)}$, with $\sum_{i=1}^{\infty} m(C_i) < \varepsilon$, $\sum_j m(D_j^{(n)}) < m(B_n)+\varepsilon < \infty$
  $\Rightarrow m^*(A_0\times B_n) \leq m^*(\bigcup_{i,j} C_i \times D_j^{(n)}) < \varepsilon\cdot(m(B_n)+\varepsilon) \to 0$.
  - each $A_n\times B_0$ is measurable, by the same reason.
$\Rightarrow A\times B$ is measurable.
- By Tonelli,
$$m(A\times B) = \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \chi_A(x)\chi_B(y)\,dx\,dy = \int_{\mathbb{R}^{d_1}} \chi_A(x)\cdot m(B)\,dx = m(A)\times m(B). \quad \square$$

· In lecture 10 we showed that if f is non-negative, measurable and a.e. finite, defined on a measurable set A with $m(A) < +\infty$, then
$$\int_A f(x)\,dx = \int_0^\infty m(\{x: f(x) \geq t\})\,dt.$$
Now we prove

Prop.: Suppose f is a measurable function on a measurable set A. Then for $\forall 1 \leq p < \infty$,
$$\int_A |f(x)|^p dx = p\int_0^\infty t^{p-1} \cdot m(\{x: |f(x)| > t\})\,dt.$$

Proof. Let $F(t,x) = \chi_{\{x: |f(x)| > t\}}$. Then by Tonelli theorem,
$$\int_A |f(x)|^p dx = \int_A \left(\int_0^{|f(x)|} p t^{p-1} dt\right) dx$$
$$= \int_A \left(\int_0^\infty p t^{p-1} \cdot F(t,x)\,dt\right) dx$$
$$= \int_0^\infty p \cdot t^{p-1}\left(\int_A F(t,x)\,dx\right) dt = p\int_0^\infty t^{p-1} \cdot m(\{x: |f(x)| > t\})\,dt \quad \square.$$

· As another application, we consider the convolution.

Def.: Given any two measurable functions f and g on $\mathbb{R}^d$, we define their <u>convolution</u> to be the function
$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy \quad \left(= \int_{\mathbb{R}^d} f(y)\,g(x-y)\,dy = (g*f)(x)\right).$$

Rmk.: The function $f(x-y)$ is measurable as a function on $\mathbb{R}^d \times \mathbb{R}^d$.
To see this

Prop.: Suppose $f, g \in L^1(\mathbb{R}^d)$. Then (1) for a.e. $x \in \mathbb{R}^d$, the convolution $f*g(x)$ exists.
(2) $f*g \in L^1(\mathbb{R}^d)$
(3) $\int_{\mathbb{R}^d} |(f*g)(x)|\,dx \leq \int_{\mathbb{R}^d} |f(x)|\,dx \cdot \int_{\mathbb{R}^d} |g(x)|\,dx.$

Proof. · We first assume $f \geq 0$, $g \geq 0$. Then by Tonelli theorem,
$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy\right) dx = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x-y)\,g(y)\,dx\right) dy$$
$$= \int_{\mathbb{R}^d} g(y)\left(\int_{\mathbb{R}^d} f(x-y)\,dx\right) dy$$
$$= \int_{\mathbb{R}^d} g(y)\left(\int_{\mathbb{R}^d} f(x)\,dx\right) dy = \int_{\mathbb{R}^d} f\,dx \cdot \int_{\mathbb{R}^d} g\,dx.$$
This proves (3) and implies (2) and (1).
· For general case, we only need the fact
$$|f| * |g|(x) \geq |(f*g)(x)|. \quad \square$$

2. Overall continuity

- Let $f \in L^p(\mathbb{R}^d)$. Then f could be "very discontinuous". However, we always have

Thm. Suppose $f \in L^p(\mathbb{R}^d)$. Then $\lim_{h\to 0}\int_{\mathbb{R}^d} |f(x+h)-f(x)|^p dx = 0$.

Proof: [Idea: To show an integral is small, you can split it into several pieces, and prove that each piece is either small by choice, or small as an integral with ⓐ bounded domain and small integrand or ⓑ small ...... bounded ......

See Lec 12, the explanation for the proof of LDCT (in measure version)]

Proof: First choose K large enough s.t. $\int_{\mathbb{R}^d \setminus B(0,K)} |f(x)|^p dx < \frac{\varepsilon}{4}\cdot\frac{1}{2^p}$.

Second find a continuous function $g \in L^p(\mathbb{R}^d)$ s.t. $\|f-g\|^p_{L^p(\mathbb{R}^d)} < \frac{\varepsilon}{8}\cdot\frac{1}{4^p}$ [We did this for $p=1$ in PSet 6, part 1, Prob 2. The case $p\geq 1$ are the same.]

Since g is continuous, it is uniformly continuous on the ball $\overline{B(0,K+2)}$.

i.e. $\exists \delta>0$ s.t. $\forall |h|<\delta$, $|g(x+h)-g(x)| < \frac{\varepsilon}{4}\cdot\frac{1}{2^p}\cdot\frac{1}{\mathrm{Vol}(B(0,K+2))}$.

$\Rightarrow \forall |h|<\delta$ (can assume $\delta<\frac{1}{2}$)

$$\int_{\mathbb{R}^d}|f(x+h)-f(x)|^p dx \leq \int_{\mathbb{R}^d\setminus B(0,K+1)} |f(x+h)-f(x)|^p dx + \int_{B(0,K+1)} |f(x+h)-f(x)|^p dx$$

$$\leq 2^p \int_{\mathbb{R}^d}\left(|f(x+h)|^p+|f(x)|^p\right)dx + \int_{B(0,K+1)}\left[2^p|g(x+h)-g(x)|^p + 4^p|f(x+h)-g(x+h)|^p + 4^p|g(x)-f(x)|^p\right]dx$$

$$< \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{4} = \varepsilon. \quad \square$$

Cor.: Suppose $f\in L^1(\mathbb{R}^d)$, g is a bounded measurable function on $\mathbb{R}^d$. Then the convolution
$$F(x) := (f*g)(x)$$
is a uniformly continuous function

Proof: Let $M = \sup_{\mathbb{R}^d}|g(x)| < +\infty$. Then

$$|F(x+h)-F(x)| = \left|\int_{\mathbb{R}^d} f(x+h-t)g(t)\,dt - \int_{\mathbb{R}^d} f(x-t)g(t)\,dt\right|$$
$$\leq \int_{\mathbb{R}^d}|f(x+h-t)-f(x-t)|\cdot|g(t)|\,dt$$
$$\leq M\cdot\int_{\mathbb{R}^d} |f(h+t)-f(t)|\,dt \longrightarrow 0 \text{ uniformly.} \quad \square$$

Cor.: Suppose $f\in L^1(\mathbb{R}^d)$, $g\in C^k(\mathbb{R}^d)$ and suppose g is compactly supported. Then $f*g\in C^k(\mathbb{R}^d)$.

(admits k^{th} derivatives which are continuous)

Proof:
- $\frac{\partial}{\partial x_i}(f*g)(x) = \frac{\partial}{\partial x_i}\int_{\mathbb{R}^d} f(y)g(x-y)\,dy = \int_{\mathbb{R}^d} f(y)\frac{\partial}{\partial x_i}g(x-y)\,dy = (f*\frac{\partial}{\partial x_i}g)(x)$ is continuous

  Use the proposition we proved last time. g is compactly support $\Rightarrow$ $\frac{\partial g}{\partial x_i}$ is bounded $\Rightarrow |f(y)\frac{\partial g}{\partial x_i}(x-y)| \leq Mf \in L^1$
- Induction. $\square$