Real Analysis

Lecture 15
04/23/2018

Last time

- Tonelli's thm: Suppose $f = f(x,y)$ is non-negative and measurable on $A\times B$, then
  (1) For a.e. $y\in B$, $f(\cdot, y)$ is measurable on A.
  (2) The function $F(y) = \int_A f(x,y)\,dx$ is measurable on B.
  (3) $$\int_{A\times B} f(x,y)\,dx\,dy = \int_A\left(\int_B f(x,y)\,dy\right)dx = \int_B\left(\int_A f(x,y)\,dx\right)dy$$

$\rightsquigarrow$
  - $m(A\times B) = m(A)\times m(B)$
  - $\int_A |f(x)|^p\,dx = p\int_0^\infty t^{p-1}\, m(\{x : |f(x)|>t\})\,dt$
  - $L : \mathbb{R}^d \to \mathbb{R}^d$ is linear $\Rightarrow m(L(A)) = |\det L|\cdot m(A)$, $\forall A\in\mathcal{L}$.
  - $f:\mathbb{R}^d\to\mathbb{R}$ non-negative, measurable $\Rightarrow \int_{\mathbb{R}^d} f(x)\,dx = m(\{(x,y) : 0\le y\le f(x)\})$.

- The convolution $f*g(x) = \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy$
  - If $f, g\in L^1(\mathbb{R}^d)$, then $f*g\in L^1(\mathbb{R}^d)$, and $\|f*g\|_{L^1}\le \|f\|_{L^1}\cdot\|g\|_{L^1}$
  - If $f\in L^1(\mathbb{R}^d)$, g is bounded measurable, then $f*g$ is uniformly continuous.
    $\leftarrow f\in L^1 \Rightarrow \lim_{h\to 0}\|f(x+h)-f(x)\|_{L^1} = 0$.
  - If $f\in L^1(\mathbb{R}^d)$, $g\in C^k(\mathbb{R}^d)$ and g is compactly supported, then $f*g\in C^k(\mathbb{R}^d)$.
    {It is enough to assume that all derivatives of g are bounded}
    - $\exists$ smooth functions of the form $f*\eta_h$ s.t. $f*\eta_h\to f$ in L^1-norm.

Today: Abstract measures

1. Measurable spaces

- We have seen that to garantee nice properties of measures (e.g. countable additivity, translation-invariance (and $m([0,1])=1$) etc.), one can't hope to "measure" all subsets.
  So to define an abstract measure on an abstract space (i.e. a set), the first thing one need to decide is: what are the sets that we ~~want to~~ must "measure"?
- To "answer" this question, let's take a closer look at Lebesgue measure.
  Let $\mathcal{L}$ = the set of Lebesgue measurable sets in $\mathbb{R}^d$. We have seen
  (1) $\emptyset\in\mathcal{L}$.
  (2) $A\in\mathcal{L}\Rightarrow A^c\in\mathcal{L}$.
  (3) $A_1, A_2\in\mathcal{L}\Rightarrow A_1\cup A_2,\ A_1\cap A_2,\ A_1\setminus A_2,\ A_1\Delta A_2\in\mathcal{L}$

(4) $A_1, A_2, \cdots \in \mathcal{L} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{L}, \quad \bigcap_{n=1}^{\infty} A_n \in \mathcal{L}$.

(5) compact sets, closed sets, open sets, F_σ sets, G_δ sets $\cdots \in \mathcal{L}$.

For abstract space, ie. a set X without any further structure, there is no conceptions like open/closed/compact sets, so we have to ignore (5) in abstract theory [But we will come back to them later.]

Also we have seen that the 2^{nd} part of (4) is a consequence of the 1^{st} part of (4) together with (2):

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c$$

Finally we notice that (3) is a consequence of (1), (2) and the 1^{st} part of (4).

(Take $A_3 = A_4 = \cdots = \phi$. Then $A_n \in \mathcal{L}$, and

$$A_1 \cup A_2 = \bigcup_{n=1}^{\infty} A_n, \quad A_1 \cap A_2 = A_1 \cap A_2 \cap A_3^c \cap A_4^c \cap \cdots$$

$$A_1 \setminus A_2 = A_1 \cap A_2^c, \quad A_1 \Delta A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1).)$$

These observations leads to

<u>Def</u>: Let X be any set. Denote $P(X) = \{A : A \subset X\}$ be the set of all subsets of X.

(1) A <u>σ-algebra</u> on X is a collection $\mathcal{F} \subset P(X)$ of subsets of X s.t.

(a) $\phi \in \mathcal{F}$

(b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(c) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

<u>Recall</u>: "σ" = countable union

(2) If $\mathcal{F}$ is a σ-algebra on X, then we call $(X, \mathcal{F})$ a <u>measurable space</u>.

According to the arguments above, we immediately get

<u>Prop.</u> Suppose $\mathcal{F}$ is a σ-algebra on X. Then

(1) $X \in \mathcal{F}$

(2) $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2,\ A_1 \cap A_2,\ A_1 \setminus A_2,\ A_1 \Delta A_2 \in \mathcal{F}$.

(3) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

<u>Rmk</u>. (1) In the definition of a σ-algebra, one can replace (c) by

(c'): $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$. ← why?

(2) If we replace (c) in the def. of σ-algebra by a weaker condition,

$(\tilde{c})$: $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2 \in \mathcal{F}$.

then the resulting collection is called a <u>Boolean algebra</u>.

- Now let's give many examples of measurable spaces.

Example 1. (Lebesgue algebra). $X = \mathbb{R}^d$, $\mathcal{F} = \mathcal{L}$.

Example 2. (Null algebra) $X = \mathbb{R}^d$, $\mathcal{F} = \{A \subset \mathbb{R}^d : m(A) = 0 \text{ or } m(A^c) = 0\}$.

Example 3. (Trivial algebra) X, $\mathcal{F} = \{\phi, X\}$.

Example 4. (Discrete algebra) X, $\mathcal{F} = P(X)$.

Example 5. (Atomic algebra) $X = \bigcup_{\alpha \in I} A_\alpha$ (non-overlapping union)

$\mathcal{F} = \{A : \exists J \subset I \text{ s.t. } A = \bigcup_{\alpha \in J} A_\alpha\}$.

[This contains Example 3, Example 4 as special cases.]

Example 6. (Elementary algebra) $X = \mathbb{R}^d$, $\mathcal{F} = \{E \subset \mathbb{R}^d : E \text{ is elementary, or } E^c \text{ is elementary}\}$.

Then $\mathcal{F}$ is a Boolean algebra on X.

BUT it is NOT a σ-algebra on X

[Similarly one can define a Jordan algebra on $\mathbb{R}^d$, which is a Boolean algebra but NOT a σ-algebra on X.]

- Prop: Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a collection of σ-algebras on X. Then $\mathcal{F} := \bigcap_\alpha \mathcal{F}_\alpha$ is also a σ-algebra on X.

Proof.
- $\phi \in \mathcal{F}_\alpha, \forall \alpha \Rightarrow \phi \in \mathcal{F}$.
- $A \in \mathcal{F} \Rightarrow A \in \mathcal{F}_\alpha, \forall \alpha \Rightarrow A^c \in \mathcal{F}_\alpha, \forall \alpha \Rightarrow A^c \in \mathcal{F}$.
- $A_1, A_2 \cdots \in \mathcal{F} \Rightarrow A_1, A_2 \cdots \in \mathcal{F}_\alpha, \forall \alpha \Rightarrow \bigcup_{n=1}^\infty A_n \in \mathcal{F}_\alpha, \forall \alpha \Rightarrow \bigcup_{n=1}^\infty A_n \in \mathcal{F}$. $\square$

Def. Let $\mathcal{X} \subset P(X)$ be any collection of subsets of X. Let

$\mathcal{F} = \langle \mathcal{X} \rangle =$ the intersection of all σ-algebras on X that contain $\mathcal{X}$.

Then $\mathcal{F}$ is a σ-algebra. We call $\mathcal{F}$ the σ-algebra generated by $\mathcal{X}$.

Rmk: - For any σ-algebra $\mathcal{F}_1$ that contains $\mathcal{X}$, one has $\langle \mathcal{X} \rangle \subset \mathcal{F}_1$

- A special example: Let $X = \mathbb{R}^d$. Then the σ-algebra generated by all open sets in X is called the Borel σ-algebra. The elements are called Borel sets.

[Of course this definition extends to any metric space, or more generally to all topological spaces.

(They are the spaces that one can define the conception of "open sets.")]

Rmk. Note that by definition, any F_σ-set is a Borel set.

We have learned that $\boxed{A \in \mathcal{L} \Leftrightarrow A = F \cup N, \text{ where } F = F_\sigma, N = \text{null}.}$

A fancy but equivalent way to say this: $\Updownarrow$

$\boxed{\text{The Lebesgue algebra } \mathcal{L} \text{ is generated by the Borel algebra and the null algebra.}}$

2. Measure spaces

- We have seen that the Lebesgue measure m is a function $m: \mathcal{L} \to [0, +\infty]$ that satisfies
  - (1) $m(\phi) = 0$.
  - (2) If $A_1, A_2, \cdots \in \mathcal{L}$ are disjoint, then $m(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty m(A_n)$.
  - (3) For any $A_1, A_2, \cdots \in \mathcal{L}$, one has $m(\bigcup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty m(A_n)$
  - (4) If $A_1 \subset A_2 \subset \cdots \in \mathcal{L}$, then $m(\bigcup_{n=1}^\infty A_n) = \lim_{n\to\infty} m(A_n)$
  - (5) If $A_1 \supset A_2 \supset \cdots \in \mathcal{L}$, and $m(A_1) < +\infty$, then $m(\bigcap_{n=1}^\infty A_n) = \lim_{n\to\infty} m(A_n)$.
  - (6) $m(A + \{x\}) = m(A)$.
  - (7) If $A_1 \subset A_2$, then $m(A_1) \le m(A_2)$
  - (8) (Fatou) If $A_1, A_2, \cdots \in \mathcal{L}$, then $m(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} m(A_n)$ (9). ......
  - (9) (Dominated Convergence) Suppose $A_n \to A$, $A_n \subset F$, $m(F) < \infty$, then $\lim_{n\to\infty} m(A_n) = m(A)$.
- Again we have to ignore (6) in the abstract setting, since the translation "$A + \{x\}$" makes no sense. [However, for those X that admits a "translation", e.g. X is a group, one does use "translation-invariant" measures.]

By carefully studying the proofs in the first part of this course, one can see that all the rest (i.e. (3), (4), (5), (7), (8)) are consequences of (1) and (2).

So we define

Def. Let $(X, \mathcal{F})$ be a measurable space.

(1) A map $\mu: \mathcal{F} \to [0, +\infty]$ is called a <u>measure</u> on $\mathcal{F}$ if

(a) $\mu(\phi) = 0$.

(b) If $A_1, A_2, \cdots \in \mathcal{F}$ are disjoint, then $\mu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$.

(2) If μ is a measure on $\mathcal{F}$, then we call $(X, \mathcal{F}, \mu)$ a <u>measure space</u>.

Rmk. Let $(X, \mathcal{F}, \mu)$ be a measure space.

(1) We say it is a <u>finite</u> measure space if $\mu(X) < +\infty$.

(2) We say it is a <u>σ-finite</u> measure space if $X = \bigcup_{n=1}^\infty A_n$, and $m(A_n) < +\infty$.

Rmk. One can define "finite-additive measure" on Boolean algebra in a similar fashion.

Prop. Let $(X, \mathcal{F}, \mu)$ be a measure space. Then

(1) If $A_1 \subset A_2 \in \mathcal{F}$, then $\mu(A_1) \le \mu(A_2)$.

(2) For any $A_1, A_2, \dots \in \mathcal{F}$, one has $\mu(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu(A_n)$.

(3) If $A_1 \subset A_2 \subset \cdots \in \mathcal{F}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

(4) If $A_1 \supset A_2 \supset \cdots \in \mathcal{F}$ and $\mu(A_1) < +\infty$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

(5) If $A_1, A_2, \dots \in \mathcal{F}$, then $\mu(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} \mu(A_n)$.

(6) ---------- and $\mu(\bigcup_{n=1}^{\infty} A_n) < +\infty$, then $\mu(\limsup_{n\to\infty} A_n) \ge \limsup_{n\to\infty} \mu(A_n)$.

(7) Suppose $A_n \to A$ (i.e. $\chi_{A_n} \to \chi_A$ pointwise), and $\exists F \in \mathcal{F}$ s.t. $A_n \subset F$ $(\forall n)$ and $\mu(F) < \infty$. Then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

Proof. Left as an exercise. $\square$

- **Examples of measures**

Example 1. (Lebesgue measure) $(\mathbb{R}^d, \mathcal{L})$, $\mu = m$.

Example 2. For any non-negative measurable function f, one can define a measure on $\mathcal{L}$ by $\mu_f(A) := \int_A f\,dx$.

Example 3. (The Dirac measure) Fix $x_0 \in X$. Then the Dirac measure δ_{x_0} at x_0 is

$$\delta_{x_0}(A) := \chi_A(x_0) = \begin{cases} 1, & x_0 \in A \\ 0, & x_0 \notin A. \end{cases}$$

Example 4. (The counting measure) For any $A \in \mathcal{F}$, one can define

$$\#A = \begin{cases} \#\text{ of elements of } A, & A \text{ is a finite set} \\ +\infty & A \text{ is NOT a finite set.} \end{cases}$$

Then one can check that $\#$ is a measure.

More generally: for any non-negative function $f : X \to [0, +\infty)$, the formula

$$\#_f(A) := \sup_{(a_n)\subset A} \sum_n f(a_n).$$

defines a measure on $(X, \mathcal{P}(X))$ (and thus on any $(X, \mathcal{F})$).

Example 5. If $(X, \mathcal{F}, \mu)$ is a measure space, and $Y \subset X$ is a subset.

- Let $\mathcal{F}_Y = \{A \cap Y \mid A \in \mathcal{F}\}$, then $(Y, \mathcal{F}_Y)$ is a measurable space.
- $Y \in \mathcal{F}$, then $(Y, \mathcal{F}_Y, \mu|_{\mathcal{F}_Y})$ is a measure space.

Example 6. Let $\mu_1, \mu_2, \dots$ are measures on $(X, \mathcal{F})$.
Then for any $c_1, c_2, \dots \ge 0$, the sum $\sum_{n=1}^{\infty} c_n \mu_n : \mathcal{F} \to [0, +\infty]$ is also a measure on $(X, \mathcal{F})$.

- A very important property of measure is completeness.

Def. Let $(X, \mathcal{F}, \mu)$ be a measure space.

(a) A set $N \in \mathcal{F}$ is called a <u>null set</u> if $\mu(N) = 0$.

(b) We say $(X, \mathcal{F}, \mu)$ is <u>complete</u> if for any null set N and for any $A \subset N$, one has $\mu(A) = 0$ (i.e. subsets of null sets are still null sets.)

[Why completeness is important? see e.g. Fubini; thm.]

e.g.: · The Lebesgue measure on $\mathcal{L}$ is complete.

· The Dirac measure on $(\mathbb{R}^d, \mathcal{L})$ is NOT complete. (why?)

Prop. Let $(X, \mathcal{F}, \mu)$ be any measure space, and $\mathcal{N} = \{N \in \mathcal{F} : \mu(N) = 0\}$. Then $\overline{\mathcal{F}} := \{A \cup B : A \in \mathcal{F}, \exists N \in \mathcal{N} \text{ s.t. } B \subset N\}$ is a σ-algebra on X. Moreover, if we define $\bar{\mu} : \overline{\mathcal{F}} \to [0, +\infty]$ by

$$\bar{\mu}(A \cup B) = \mu(A), \quad \forall A \in \mathcal{F},\ B \subset N \in \mathcal{N}.$$

then $\bar{\mu}$ is a complete measure on $(X, \overline{\mathcal{F}})$ that extends μ, i.e. $\bar{\mu} = \mu$ on $\mathcal{F}$

The proof is a straightforward computation and thus we omit it.

Rmk: One can regard the Lebesgue measure m as the complete of the (finite additive) measure of the Borel measure space.