Real Analysis

Last time:

- A measurable space is a pair $(X, \mathcal{F})$, where X is a set, $\mathcal{F}$ is a σ-algebra on X:
  (a) $\phi \in \mathcal{F}$
  (b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
  (c) $A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

  $\Rightarrow$ $A \cup B$, $A \cap B$, $A \setminus B$, $A \triangle B$, $\bigcap_{n=1}^{\infty} A_n$, $\limsup_{n\to\infty} A_n$, $\liminf_{n\to\infty} A_n$ $\in \mathcal{F}$.

- A measure space is a triple $(X, \mathcal{F}, \mu)$, where $(X, \mathcal{F})$ is a measurable space and $\mu: \mathcal{F} \to [0, +\infty]$ is a measure, i.e.
  (i) $\mu(\phi) = 0$
  (ii) If $A_1, A_2, \cdots \in \mathcal{F}$ are disjoint, then $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$.

  ($\Rightarrow$ monotonicity, sub-additivity, MC, Fatou, DC etc.)

Today: Integration on a measure space

1. Measurable functions.

- Let $(X, \mathcal{F})$ be a measurable space.

  As in the Lebesgue theory we can define

  Def. A function $f: X \to \mathbb{R}$ (or $[-\infty, +\infty]$) is measurable if for any $t \in \mathbb{R}$, the set $\{x \in X : f(x) > t\} \in \mathcal{F}$.

  As in Lec. 6 and PSet 3, part 2, one can prove

  Prop. f is measurable $\Leftrightarrow$ $\{x \in X : f(x) > t\} \in \mathcal{F}$, $\forall t$
  $\Leftrightarrow$ $\{ \text{------} \geq \text{------}$
  $\Leftrightarrow$ $\text{------} < \text{------}$
  $\Leftrightarrow$ $\text{------} \leq \text{------}$
  $\Leftrightarrow$ $\forall$ open set $U \subset \mathbb{R}$, $f^{-1}(U) \in \mathcal{F}$
  $\Leftrightarrow$ ... close ------
  $\Leftrightarrow$ ... Borel ------

Rmk: In general, given two measurable spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, we say a map $\varphi: (X, \mathcal{F}) \to (Y, \mathcal{G})$ is measurable if $\forall B \in \mathcal{G}$, one has $\varphi^{-1}(B) \in \mathcal{F}$.

So a measurable function is a measurable map $f: (X, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$. ($\mathcal{B}$ = the Borel algebra.)

[In particular, a measurable function f on $\mathbb{R}$ is a measurable map $f: (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{B})$, not a measurable map $(\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{L})$!]

The next proposition shows that almost all nice properties in Lebesgue theory are still valid for measurable functions on abstract measurable spaces.

Prop. Let $(X, \mathcal{F})$ be a measurable space.

(1) If f, g are $\mathbb{R}$-valued (or $[0,+\infty]$-valued) measurable functions, so are cf, $f+g$, fg.

(2) If $\varphi: \mathbb{R} \to \mathbb{R}$ is continuous, $f: X \to \mathbb{R}$ is measurable, then $\varphi \circ f: X \to \mathbb{R}$ is measurable.

(3) If f_n are a sequence of measurable functions, $f_n \to f$, then f is measurable.

More generally: $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$, $\sup_n f_n$, $\inf_n f_n$

(0) $A \in \mathcal{F} \Leftrightarrow \chi_A$ is measurable.

Proof: Exercise. $\square$

- We can also define simple functions using which we will develop the theory of integration.

Def. A <u>simple function</u> on a measurable space $(X, \mathcal{F})$ is a function of the form
$$f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$$
where $A_i \in \mathcal{F}$, and $a_i \in \mathbb{R}$ (or $a_i \in [0,+\infty]$).

Prop. Let $(X, \mathcal{F})$ be a measurable space, and $f: X \to [0,+\infty]$ a non-negative measurable function. Then there exists a monotone sequence of simple functions
$$0 \le f_1 \le f_2 \le \cdots \nearrow f.$$
Moreover, the convergence is uniform on any set on which f is bounded.

Proof: Repeat the proof in Lec. 8, (page 2), i.e. let
$$f_n(x) = \begin{cases} \frac{k-1}{2^n}, & \text{if } \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n} \quad (k = 1, 2, \cdots, n \cdot 2^n) \\ n, & \text{if } f(x) \ge n \end{cases}$$
then check ① $0 \le f_{n+1}(x) - f_n(x) \le \frac{1}{2^{n+1}}$ if $f(x) < n$

② $0 \le f_{n+1}(x) - f_n(x) \le 1$ if $f(x) \ge n$.

and then the result follows. $\square$

2. Integration: Non-negative theory

- Now let $(X, \mathcal{F}, \mu)$ be a measure space.

For any non-negative simple function $g = \sum_{i=1}^{n} a_i \chi_{A_i} : X \to [0, +\infty]$, we can define

$$\int_X g \, d\mu := \sum_{i=1}^{n} a_i \mu(A_i).$$

[Of course one still need to check that different representatives of f gives the same result.]

Once again, we can ~~easy~~ easily prove the properties of integrals of simple functions as we did in Lecture 9, and thus define

Def: Let $f : X \to [0, +\infty]$ be measurable. Then

$$\int_X f \, d\mu = \sup\left\{ \int_X g \, d\mu : 0 \le g \le f, \ g \text{ is } \mu\text{measurable simple} \right\}.$$

The basic properties like monotonity is again a direct consequence.

To prove linearity, again one need to prove

Thm (Monotone Convergence Theorem) Let $(X, \mathcal{F}, \mu)$ be a measure space, and

$$0 \le f_1 \le f_2 \le \cdots$$

be an increasing sequence of measurable functions. Then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu.$$

[The proof is word-by-word the same as before.]

As corollaries, one has: For non-negative measurable functions,

(1) linearity: $\int_X (c_1 f_1 + c_2 f_2) \, d\mu = c_1 \int_X f_1 \, d\mu + c_2 \int_X f_2 \, d\mu, \quad c_1, c_2 > 0.$

(2) countable additivity I: $\int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$

(3) — — — — — — — II: $\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu$, where A_n's are disjoint

(4) Fatou's lemma: $\int \liminf_{n\to\infty} f_n \, d\mu = \liminf_{n\to\infty} \int f_n \, d\mu.$

Is there anything new? NO!

Note: The countable additivity II $\Rightarrow$ $\mu_f = f d\mu$ is a measure!

Prop (Countable additivity III) Let $(X, \mathcal{F})$ be measurable space, $f : X \to [0, +\infty]$ be a non-negative measurable function, and $\mu_1, \mu_2, \cdots$ be a sequence of measures on $\mathcal{F}$. Then

$$\int_X f \, d\sum_{n=1}^{\infty} \mu_n = \sum_{n=1}^{\infty} \int_X f \, d\mu_n.$$

Proof: If $f = \chi_A$, this is obvious since $LHS = \sum_{n=1}^{\infty} \mu_n(A) = RHS$.

By linearity, this is true for simple functions.

By MCT, this is true for any non-negative measurable function. □

Rmk: Consider the counting measure $\#$ on $\{1, 2, \cdots\}$.

Let a_n be a sequence of non-negative numbers.

Then we can view $\{a_n\}$ as a function $a: \{1, 2, 3, \cdots\} \to [0, +\infty]$.

By definition,
$$\int_{\mathbb{N}} a \, d\# = \sum_{n=1}^{\infty} a_n .$$

At beginning of Lec. 9, we mentioned "infinite sums are discrete version of integrals"

Now we can make this more precise:

$$\boxed{\text{Infinite sums are integrals w.r.t. the counting measure}}.$$

In particular, if you regard $\{f_n\}$ as
$$f: X \times \mathbb{N} \to [0, +\infty]$$
$$f(x, n) \mapsto f_n(x).$$

Then countable additivity I is
$$\int_X \left(\int_{\mathbb{N}} f(x, n) \, d\#_n \right) d\mu_x = \int_{\mathbb{N}} \left(\int_X f(x, n) \, d\mu_x \right) d\#_n$$
which is Tonelli's theorem!

3. Integration. Absolute convergent theory

· **Def**: Let f be any measurable $\mathbb{R}$-valued function on measure space $(X, \mathcal{F}, \mu)$.

We say f is (absolutely) integrable if
$$\|f\|_{L^1(X, \mathcal{F}, \mu)} := \int_X |f| \, d\mu < +\infty$$

For such f, we define
$$\int_X f \, d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

Properties like monotonicity, linearity, countable additivity (absolute continuity) extends to this setting without any difficulty.

Following Lecture 12 one can prove

Thm. (Dominated Convergence Theorem)

Suppose (1) $f_n \to f$ a.e. (or in measure)

$\exists g \in L^1(X, \mathcal{F}, \mu)$ s.t. $|f_n| \le g$ a.e. $\Rightarrow$ · $f \in L^1$ · $\|f_n - f\|_{L^1} \to 0$

As consequences, one can prove: ~~countable additivity~~ etc..

· completeness of L^p

·

- So: what could be different?

The ~~[illegible]~~ theorems of Tonelli and Fubini! We used "completeness" of the Lebesgue measure.

In last problem set (PSet 8-1), we introduced the product of σ-algebras:

$(X, \mathcal{F})$, $(Y, \mathcal{G}) \longrightarrow (X\times Y, \mathcal{F}\otimes\mathcal{G})$,

where $\mathcal{F}\otimes\mathcal{G}$ is generated by sets of the form $A\times B$, $A\in\mathcal{F}$, $B\in\mathcal{G}$.

<u>Fact</u>: Suppose $(X, \mathcal{F}, \mu_X)$ and $(Y, \mathcal{G}, \mu_Y)$ be two σ-finite measure spaces. Then there exists a unique measure $\mu_X\times\mu_Y$ on $\mathcal{F}\otimes\mathcal{G}$, s.t.
$$(\mu_X\times\mu_Y)(A\times B) = \mu_X(A)\cdot\mu_Y(B), \quad \forall A\in\mathcal{F},\ B\in\mathcal{G}.$$

The proof is non-trivial, and we will postpone the proof to next time.

<u>Thm</u> (Tonelli's theorem: incomplete version)

Let $(X, \mathcal{F}, \mu_X)$ and $(Y, \mathcal{G}, \mu_Y)$ be σ-finite measure spaces, and $f: X\times Y\to[0,+\infty]$ be a $\mathcal{F}\otimes\mathcal{G}$ measurable function.

Then (1) The functions $f(\cdot, y)$ is measurable w.r.t. μ_X

(2) The function $y\longmapsto\int_X f(x,y)\,d\mu_X$ is measurable w.r.t. μ_Y

(3) $\int_{X\times Y} f(x,y)\,d\mu_X\times d\mu_Y = \int_Y\left(\int_X f(x,y)\,d\mu_X\right)d\mu_Y = \int_X\left(\int_Y f(x,y)\,d\mu_Y\right)d\mu_X$.

Let $X=[0,1]$, $\mathcal{F}=\mathcal{P}(X)$, $\mu=\#$ ← NOT σ-finite.
$Y=[0,1]$, $\mathcal{F}=\mathcal{L}$, $\mu=m$.
Then for $f(x,y)=\begin{cases}1, & x=y\\ 0, & x\neq y\end{cases}$ one has
$\int_X\left(\int_Y f(x,y)\,dm_y\right)d\#_x = \int_X 0\,d\#_x = 0$
$\int_Y\left(\int_X f(x,y)\,d\#_x\right)dm_y = \int_Y 1\,dm_y = 1$.

<u>Idea</u>: • σ-finite $\Rightarrow X=\bigcup_{n=1}^{\infty}X_n$, $Y=\bigcup_{n=1}^{\infty}Y_n$.

So by applying monotone convergence, one may assume $\mu_X(X)<\infty$, $\mu_Y(Y)<\infty$.

- By linearity and simple function approximation, one may assume $f=\chi_S$, $S\in\mathcal{F}\otimes\mathcal{G}$.
- Show that if $S=(A_1\times B_1)\sqcup\cdots\sqcup(A_k\times B_k)$, then χ_S satisfies the theorem.
- Applying MCT many times to prove the theorem. $\square$

- This is different from Lebesgue theory, because

The Lebesgue measure $m^{k_1+k_2}$ on $\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}$ is NOT the product $m^{k_1}\times m^{k_2}$

the Lebesgue measure on $\mathbb{R}^{k_1}$

<u>Reason</u>: ~~[illegible]~~ For the σ-algebras,
$$\mathcal{L}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}) \neq \mathcal{L}(\mathbb{R}^{k_1})\otimes\mathcal{L}(\mathbb{R}^{k_2}).$$

In fact, $(\mathbb{R}^{k_1+k_2}, \mathcal{L}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}), m^{k_1+k_2})$ is the <u>completion</u> of $(\mathbb{R}^{k_1}, \mathcal{L}(\mathbb{R}^{k_1}), m^{k_1})\times(\mathbb{R}^{k_2}, \mathcal{L}(\mathbb{R}^{k_2}), m^{k_2})$

$\rightsquigarrow$ complete version of Tonelli's theorem and Fubini's theorem.

<u>Reason</u>: According to PSet 8-1, Problem 4, part (1)

if $E\in\mathcal{L}(\mathbb{R})\otimes\mathcal{L}(\mathbb{R})$, then $\forall x\in\mathbb{R}$, the set $E_x:=\{y\in Y: (x,y)\in E\}\in\mathcal{L}(\mathbb{R})$

So if we let $E=\{(0,y)\mid y\in\tilde{A}\}$, where $\tilde{A}\subset\mathbb{R}$ is not measurable,

then $E\notin\mathcal{L}(\mathbb{R})\otimes\mathcal{L}(\mathbb{R})$. But obvious $E\in\mathcal{L}(\mathbb{R}^2)$