Real Analysis

Lecture 17
04/28/2018

Last time

- Measurable function. $f: X \to \mathbb{R}$. $\Longleftrightarrow$ measurable map $f: (X, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$.
  - many properties in Lebesgue theory still holds.
- Integration: non-negative theory: same definition, same properties (MCT, Fatou etc.)
  - Three "countable additivity", corresponds to three elements of an integral (domain, function, measure)
  - infinite sum = integral w.r.t. the counting measure
- Integration: absolute convergence theory: same definition, same properties (continuity, DCT)
  - Tonelli, Fubini could have different forms.
    depends on the fact whether you are using $\mu_X \times \mu_Y$ or $\overline{\mu_X \times \mu_Y}$ for the product.
    this is easier (for $\mu_X \times \mu_Y$); this is the same as the Lebesgue theory (for $\overline{\mu_X \times \mu_Y}$).

    missing, existence of product measure on product space

Today: Construct measures

1. Outer measure

- We studied abstract def. of measure.
  Given a measure space, we studied the integration theory.
  But... how do we get a measure μ on a measurable space? [In particular, how do we get the σ-algebra $\mathcal{F}$?]

  Recall: In Lebesgue theory, we started with

  elementary measure $\rightsquigarrow$ Jordan outer measure $\xrightarrow{\text{premeasure extension}}$ Lebesgue outer measure m^*
  $\downarrow$ Caratheodory
  Lebesgue measure m

  The difference between m^* and m:

  (a) m^* satisfies countable subadditivity, while m is countable additive.

  (b) m^* is defined for all sets in $P(\mathbb{R}^d)$, while m is only defined for sets in $\mathcal{L}$.

  By staring at the properties of Lebesgue outer measure, one can define

Def: Let X be a set. An outer measure is a map $\mu^*: P(X) \to [0, +\infty]$ such that
(i) $\mu^*(\emptyset) = 0$.
(ii) If $A_1 \subset A_2$, then $\mu^*(A_1) \le \mu^*(A_2)$ $\leftarrow$ monotonicity
(iii) If $A_1, A_2, \cdots$, then $\mu^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$ $\leftarrow$ countable subadditivity

Note: non-negativity + additivity $\Rightarrow$ monotonicity
non-negativity + subadditivity $\not\Rightarrow$ monotonicity.

We will see that it is much easier to get outer measures than measures

• **Example** (Prop.): Let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of subsets of X s.t. $\phi \in \mathcal{E}$, $\bigcup_{E\in\mathcal{E}} E = X$.

Given any function $\rho: \mathcal{E} \to [0,+\infty]$ with $\rho(\phi)=0$, one can define

$$\mu^*(A) = \inf\left\{ \sum_n \rho(E_n) : A \subset \bigcup_{n=1}^{\infty} E_n,\ E_n \in \mathcal{E} \right\}.$$

<u>Fact</u>: μ^* is an outer measure.

<u>Proof</u>: · It is obvious that $\mu^*(\phi)=0$ and $\mu^*(A_1) \le \mu^*(A_2)$ for $A_1 \subset A_2$.

· If there exists A_n s.t. $\mu^*(A_n) = +\infty$, then obvious $\mu^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$.

Now suppose $\mu^*(A_n) < +\infty$ for all n.

Then we take $E_m^{(n)}$ s.t. $A_n \subset \bigcup_m E_m^{(n)}$, $\mu^*(A_n) \ge \sum_m \rho(E_m^{(n)}) - \frac{\varepsilon}{2^n}$. $\boxed{\varepsilon\text{-trick!}}$

$\Rightarrow \bigcup_n A_n \subset \bigcup_{n,m} E_m^{(n)}$.

$\Rightarrow m^*(\bigcup_n A_n) \le \sum_{n,m} \rho(E_m^{(n)}) \le \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon$.

Letting $\varepsilon \to 0$, we get countable subadditivity. $\square$

• So there are plenty of outer measures.

How do we get measures (and in particular $\mathcal{F}$) from an outer measure?

<u>Recall</u>: In Lebesgue theory,

$$\begin{aligned} A \in \mathcal{L} &\iff \forall \varepsilon>0,\ \exists \text{ open set } U \supset A \text{ s.t. } m^*(U\setminus A) < \varepsilon \\ &\iff \forall \varepsilon > 0,\ \exists \text{ closed set } F \subset A \text{ s.t. } m^*(A \setminus F) < \varepsilon \\ &\iff A = G \setminus N,\quad G = G_\delta,\ N = \text{Null} \\ &\iff A = F \cup N,\quad F = F_\sigma,\ N = \text{Null} \\ &\iff \forall \text{ box } B,\quad |B| = m^*(B\cap A) + m^*(B\setminus A) \\ &\iff \forall T \subset \mathbb{R}^d,\quad m^*(T) = m^*(T\cap A) + m^*(T\setminus A). \leftarrow \text{Caratheodory} \end{aligned}$$

For an abstract, there is no conception of open, closed, F_σ, G_δ, and box!

So we only have one choice.

<u>Def</u>: Let μ^* be an outer measure on X. A set $A \subset X$ is <u>(Caratheodory) measurable</u> w.r.t. μ^* if one has

$$\mu^*(T) = \mu^*(T\cap A) + \mu^*(T \setminus A), \qquad \forall T \subset X.$$

<u>Rmk</u>: • By subadditivity, one always has $\mu^*(T) \le \mu^*(T\cap A) + \mu^*(T\setminus A)$.

• If $\mu^*(A) = 0$, then A is Caratheodory measurable w.r.t. μ^*, since

$$\mu^*(T) \underset{\substack{\uparrow \\ \text{monotonicity}}}{\ge} \mu^*(T\setminus A) = \mu^*(T\setminus A) + \mu^*(T\cap A)$$

$\mu^*(T\cap A) \le \mu^*(A)$
$\Rightarrow \mu^*(T\cap A) = 0$.

· The main theorem is

Thm. (Caratheodory extension theorem) Suppose $\mu^*: \mathcal{P}(X) \to [0,+\infty]$ is any outer measure.
Let $\mathcal{F} = \{A \in \mathcal{P}(X) : A$ is Caratheodory measurable w.r.t. $\mu^*\}$.
Then $\mathcal{F}$ is a σ-algebra, and $\mu = \mu^*|_{\mathcal{F}} : \mathcal{F} \to [0,+\infty]$ is a measure.

Proof (1). obviously $\phi \in \mathcal{F}$.

· If $A \in \mathcal{F}$, i.e. $\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \setminus A) = \mu^*(T\cap A) + \mu^*(T \cap A^c)$,
~~then~~ obviously $A^c \in \mathcal{F}$.

· Now suppose $A_1, A_2, \cdots \in \mathcal{F}$.

- First we show $A_1 \cup A_2 \in \mathcal{F}$, i.e. $\mu^*(T) \geq \mu^*(T \cap (A_1 \cup A_2)) + \mu^*(T \setminus (A_1 \cup A_2))$.
To see this, we just notice

$$\begin{aligned}\mu^*(T) &= \mu^*(T\cap A_1) + \mu^*(T \cap A_1^c)\\ &= \mu^*(T\cap A_1 \cap A_2) + \mu^*(T \cap A_1 \cap A_2^c) + \mu^*(T \cap A_1^c \cap A_2) + \mu^*(T\cap A_1^c \cap A_2^c)\\ &\geq \mu^*(T \cap ((A_1\cap A_2) \cup (A_1 \cap A_2^c) \cup (A_1^c \cap A_2))) + \mu^*(T \cap (A_1 \cup A_2)^c)\\ &= \mu^*(T \cap (A_1 \cup A_2)) + \mu^*(T \cap (A_1 \cup A_2)^c).\end{aligned}$$

- To prove $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, we write $B = \bigcup_{n=1}^{\infty} A_n$, and $B_N = \bigcup_{n=1}^{N} A_n$.
Then by induction, $B_N \in \mathcal{F}$ for all N. So we have

$$\mu^*(T) = \mu^*(T \cap B_N) + \mu^*(T \cap B_N^c)$$

__Note__: · $\mu^*(T \cap B_N^c) \geq \mu^*(T \cap B^c) \Rightarrow \lim_{N\to\infty} \mu^*(T \cap B_N^c) \geq \mu^*(T \cap B^c)$

· $$\begin{aligned}\mu^*(T \cap B_{N+1}) &= \mu^*(T \cap B_{N+1} \cap B_N) + \mu^*(T \cap B_{N+1} \cap B_N^c)\\ &= \mu^*(T \cap B_N) + \mu^*(T \cap (B_{N+1} \setminus B_N))\end{aligned}$$

$$\Rightarrow \lim_{N\to\infty} \mu^*(T\cap B_N) = \sum_{N=0}^{\infty} \mu^*(T \cap (B_{N+1} \setminus B_N))$$

So by letting $N \to \infty$, we get

$$\mu^*(T) \geq \sum_{N=0}^{\infty} \mu^*(T \cap (B_{N+1} \setminus B_N)) + \mu^*(T\cap B^c) \geq \mu^*(T \cap B) + \mu^*(T \cap B^c).$$

This implies $B = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, i.e. $\mathcal{F}$ is a σ-algebra

(2) It remains to show that $\mu = \mu^*|_{\mathcal{F}}$ is a measure.
Obviously $\mu(\phi) = \mu^*(\phi) = 0$
Now let $A_1, A_2, \cdots \in \mathcal{F}$ be disjoint. Then

- $\mu(A_1 \cup A_2) = \mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*((A_1 \cup A_2) \cap A_1^c) = \mu^*(A_1) + \mu^*(A_2)$.
- By induction, $\mu(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n)$.

$\Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) \geq \mu(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n) \longrightarrow \sum_{n=1}^{\infty} \mu(A_n)$

$\Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. $\square$

Rmk: If $\mu^*(A)=0$, $B\subset A$, then $\mu^*(B)=0$. So $B\in\mathcal{M}$ and $\mu(B)=0$.
In other words:
the measure μ constructed by Caratheodory' extension then is complete!

2. Premeasure.

Recall: Lebesgue outer measure was constructed via Jordan measure (or elementary measure), which are finitely additive on Boolean algebras.

Question: Given any Boolean algebra $\mathcal{B}$ and a finitely additive measure $\mu_0: \mathcal{B}\to[0,+\infty]$, can one "refine" $\mathcal{B}$ to a σ-algebra $\mathcal{M}$ and extend μ_0 to a countable additive measure μ? Of course we need:

Def: A premeasure μ_0 on a Boolean algebra $\mathcal{B}$ is a finitely additive measure $\mu_0: \mathcal{B}\to[0,+\infty]$ with the additional property that if $A_1, A_2, \cdots\in\mathcal{B}$ disjoint, and $\bigcup_{n=1}^\infty A_n\in\mathcal{B}$, then $\mu_0(\bigcup_{n=1}^\infty A_n)=\sum_{n=1}^\infty\mu_0(A_n)$.

[It is NOT a measure because still possible that $A_1, A_2, \cdots\in\mathcal{B}$, $\bigcup_{n=1}^\infty A_n\notin\mathcal{B}$.]

Example: The elementary measure on the elementary Boolean algebra is a premeasure.

Example: Let $(X,\mathcal{M},\mu_X)$ and $(Y,\mathcal{N},\mu_Y)$ be measure spaces.

Let $\mathcal{B}=\{(A_1\times B_1)\cup\cdots\cup(A_k\times B_k) : A_i\in\mathcal{M}, B_i\in\mathcal{N}\}$.

Then $\mathcal{B}$ is a Boolean algebra.

[To check $S\in\mathcal{B}\Rightarrow S^c\in\mathcal{B}$, one only need to observe $\cdot(A\times B)^c=(A^c\times B)\cup(A\times B^c)\cup(A^c\times B^c)$
$\cdot(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$ - -]

Define $\mu_0:\mathcal{B}\to[0,+\infty]$ by
$$\mu_0((A_1\times B_1)\cup\cdots\cup(A_k\times B_k))=\sum_{i=1}^k\mu_X(A_i)\cdot\mu_Y(B_i)$$
where the union is a disjoint union. [One can ALWAYS rewrite ~~make~~ to union to a disjoint union.] (need to check well-definedness)

One can check (NOT easy. Need to apply MCT of integrals of non-negative functions) that μ_0 is a premeasure.

e.g. for the simple case $A\times B=\sum_{k=1}^\infty A_k\times B_k$, one has
$$\chi_A(x)\chi_B(y)=\sum_{k=1}^\infty\chi_{A_k}(x)\chi_{B_k}(y)$$
$$\Rightarrow\int_Y\chi_A(x)\chi_B(y)dy=\int_Y\sum_{k=1}^\infty\chi_{A_k}(x)\chi_{B_k}(y)dy$$
$$\Rightarrow\chi_A(x)\mu_Y(B)=\sum_{k=1}^\infty\chi_{A_k}(x)\mu_Y(B_k)$$
$$\Rightarrow\mu_X(A)\mu_Y(B)=\sum_{k=1}^\infty\mu_X(A_k)\mu_Y(B_k).$$

<u>Example</u>: Not all finitely additive measures on a Boolean algebra are premeasures.

<u>e.g.</u> $X = \mathbb{N}$, $\mathcal{B} = \mathcal{P}(X)$, $\mu_0(A) = \begin{cases} 0, & \text{if } A \text{ is finite} \\ \infty, & \text{if } A \text{ is not finite}. \end{cases}$

Then A is finitely additive, but not a premeasure!

Now we prove

<u>Thm.</u> (Hahn-Kolmogorov) Let $\mathcal{B}$ be a Boolean algebra on X and $\mu_0 : \mathcal{B} \to [0, +\infty]$ is a premeasure. Then one can extend $\mathcal{B}$ to a σ-algebra $\mathcal{J}_c$ on X, and μ_0 to a countably additive measure $\mu : \mathcal{J}_c \to [0, +\infty]$.

<u>Proof.</u> We have seen that

(a) $\mu^*(A) := \inf \{ \sum_n \mu_0(E_n) : A \subset \bigcup_{n=1}^{\infty} E_n,\ E_n \in \mathcal{B} \}$

is an outer measure

(b) $\mathcal{J}_c = \{ A \in \mathcal{P}(X) : A$ ~~satisfies the~~ is Caratheodory measurable w.r.t. $\mu^* \}$

is a σ-algebra, and $\mu = \mu^*|_{\mathcal{J}_c}$ is a (complete) measure.

It remains to prove (1) $\mathcal{J}_c$ refines $\mathcal{B}$ (i.e. $\mathcal{B} \subset \mathcal{J}_c$)

(2) $\mu = \mu_0$ on $\mathcal{B}$.

<u>(1):</u> Let $A \in \mathcal{B}$. We need to show $\mu^*(T) \geq \mu^*(T \cap A) + \mu^*(T \setminus A)$.

We may assume $\mu^*(T) < +\infty$. Apply ε-trick:

By def. of μ^*, $\exists E_1, E_2, \cdots \in \mathcal{B}$ s.t. $T \subset \bigcup_{n=1}^{\infty} E_n$, $\sum_{n=1}^{\infty} \mu_0(E_n) \leq \mu^*(T) + \varepsilon$.

Since $\mathcal{B}$ is a Boolean algebra, $A \cap E_n \in \mathcal{B}$. So

$$T \cap A \subset \bigcup_n (A \cap E_n) \Rightarrow \mu^*(T \cap A) \leq \sum_{n=1}^{\infty} \mu_0(A \cap E_n)$$

Similarly

$$\mu^*(T \setminus A) = \mu^*(T \cap A^c) \leq \sum_{n=1}^{\infty} \mu_0(E_n \cap A^c) = \sum_{n=1}^{\infty} \mu_0(E_n \setminus A).$$

Since $\mu_0(A \cap E_n) + \mu_0(E_n \setminus A) = \mu_0(E_n)$, we get

$$\mu^*(T \cap A) + \mu^*(T \setminus A) \leq \sum_{n=1}^{\infty} \mu_0(E_n) \leq \mu^*(T) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get $\mu^*(T) \geq \mu^*(T \cap A) + \mu^*(T \setminus A)$, i.e. $A \in \mathcal{J}_c$.

<u>(2).</u> Let $A \in \mathcal{B}$. We need to show $\mu^*(A) = \mu_0(A)$.

By def., we only need to show $\mu^*(A) \geq \mu_0(A)$

Suppose $E_1, E_2, \cdots \in \mathcal{B}$, $A \subset \bigcup_{n=1}^{\infty} E_n$. Let $\tilde{E}_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$.

Then $\tilde{E}_n \in \mathcal{B}$, $A \subset \bigcup_{n=1}^{\infty} \tilde{E}_n$, and they are disjoint.

Since μ_0 is a premeasure, we get

$$\mu_0(A) = \mu_0\Big(\bigcup_{n=1}^{\infty} (\tilde{E}_n \cap A)\Big) = \sum_{n=1}^{\infty} \mu_0(\tilde{E}_n \cap A) \leq \sum_{n=1}^{\infty} \mu_0(E_n)$$

Taking infimum, we get $\mu_0(A) \leq \mu^*(A)$. $\square$

<u>Cor</u>: Existence of product measure $\mu_X \times \mu_Y$ on $X \times Y$.