Real Analysis

Lecture 18
05/03/2018

Last time

- Outer measure: $\mu^*: P(X) \to [0,+\infty]$ s.t. ① $\mu^*(\phi)=0$
  ② $\mu^*(A_1) \le \mu^*(A_2)$ for $A_1 \subseteq A_2$
  ③ $\mu^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$.

Caratheodory measurability. A is measurable w.r.t. $\mu^* \Leftrightarrow \mu^*(T) = \mu^*(T\cap A) + \mu^*(T \cap A^c)$

Premeasure: $\mu_0 : \mathcal{B} \to [0,+\infty]$ is a finitely additive measure on a Boolean algebra ~~which is~~ "with no obstruction of being a measure", i.e. if $A_1, A_2, \cdots \in \mathcal{B}$, disjoint, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$, then $\mu_0(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu_0(A_n)$.

- To construct a measure:
  Step 1: Start with any collection $\mathcal{E} \subset P(X)$ and any function $\rho: \mathcal{E} \to [0,+\infty]$, $\rho(\phi)=0$.
  Step 2: Construct an outer measure $\mu^*(A) := \inf\{\sum_{n=1}^{\infty} \rho(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n,\ A_n \in \mathcal{E}\}$
  Step 3: Construct a σ-algebra $\mathcal{F} = \{A \in P(X) : A$ is Caratheodory measurable w.r.t. $\mu^*\}$.
  Then $\mu := \mu^*|_{\mathcal{F}}$ is a complete measure on $\mathcal{F}$.

To get a reasonably nice measure:
  Step 1': Instead of arbitrary $(\mathcal{E}, \rho)$ above, you start with a premeasure $(\mathcal{B}, \mu_0)$.

Today. Metric v.s. measure

1. Metric outer measure

- Now we would like to take a closer look at the role of open/closed sets in Lebesgue theory.

To extend the corresponding part to more general setting, we need a "topology" on the abstract set X, so that we can talk about "open", "closed".

For simplicity, we will assume X admit a "structure of distance".

Def: Let X be a set. A distance on X is a function $d: X \times X \to [0,+\infty)$ s.t. (a) $d(x,y)=0 \Leftrightarrow x=y$.
(b) $d(x,y)=d(y,x)$, $\forall x,y \in X$. (symmetry)
(c) $d(x,z) \le d(x,y)+d(y,z)$, $\forall x,y,z \in X$. (triangle inequality)

We say (X,d) is a metric space.

Example: $X = \mathbb{R}^d$, $\quad d(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_d - y_d)^2}$.

Example: Any X, $\quad d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$.

Example: $X = L^1(\mathbb{R}^d)$ (where $f = g$ a.e. means f and g are the same element in $L^1(\mathbb{R}^d)$)

$$d(f,g) = \int_{\mathbb{R}^d} |f-g|\,dx.$$

- Now let (X,d) be a metric space.

  Then we can talk about open balls.

  $$B_r(x) = \{y \in X : d(x,y) < r\}.$$

  $\rightsquigarrow$ open set $U \Leftrightarrow \forall x \in U, \exists r > 0$ s.t. $B_r(x) \subset U$.

  $\rightsquigarrow$ closed set $F \Leftrightarrow F^c$ is open $\rightsquigarrow$ (compact = ~~bounded closed~~ any open covering admits a finite sub-covering)

  $\rightsquigarrow$ F_σ set, G_δ set, Borel sets

  $\rightsquigarrow$ Borel σ-algebra $\mathcal{B}_X$ = the σ-algebra generated by open sets (or closed sets or compact sets)

  Def: A measure on $\mathcal{B}_X$ is called a Borel measure.

- Given any two sets A, B in a metric space (X,d), we can define the "distance"

  $$d(A,B) := \inf\{d(x,y) : x \in A, y \in B\}.$$

  Rmk: • This is NOT a distance function on the set $\mathcal{P}(X)$.

  In fact, by def., if $A \cap B \neq \emptyset$, then we always has $d(A,B) = 0$.

  •  $d(A,C) > d(A,B) + d(B,C)$.

  • There is a distance function on $\mathcal{K}(X) = \{$all compact sets in $X\}$:

  $$d_H(A,B) = \inf\{\delta : A \subset B_\delta(B),\ B \subset B_\delta(A)\}. \quad \leftarrow \text{Hausdorff distance}.$$

- We have seen in PSet 2, Part 1 that the Lebesgue outer measure m^* satisfied

  $$d(A,B) > 0 \Rightarrow m^*(A \cup B) = m^*(A) + m^*(B)$$

  and we used this property in Lec. 4 to show that any compact set is Lebesgue measurable ($\rightsquigarrow$ Any Borel set in $\mathbb{R}^d$ is Lebesgue measurable).

  Def: An outer measure μ^* on a metric space (X,d) is called a metric outer measure if

  $$d(A,B) > 0 \Rightarrow \mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Thm. Suppose μ^* is a metric outer measure on a metric space (X, d). Then all Borel sets are measurable w.r.t. μ^*.
(So $\mu := \mu^*|_{\mathcal{B}_X}$ is a Borel measure.)

Proof: • Since all μ^*-measurable sets form a σ-algebra, it is enough to prove that any closed set F in X is measurable.

WLOG, let $T \subset X$ be any set s.t. $\mu^*(T) < +\infty$.

Then what we need to prove is $\mu^*(T) \geq \mu^*(T \cap F) + \mu^*(T \setminus F)$.

• For each n, we let $T_n = \{x \in T\setminus F : d(x, F) \geq \frac{1}{n}\}$.

Since F is closed, we have $T \setminus F = \bigcup_{n=1}^{\infty} T_n$.

By def., $d(T_n, F) \geq \frac{1}{n} > 0$. Since μ^* is a metric outer measure,

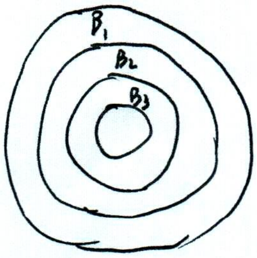
$$\mu^*(T) \geq \mu^*((T\cap F) \cup T_n) = \mu^*(T \cap F) + \mu^*(T_n). \quad (*)$$

Claim: $\lim_{n\to\infty} \mu^*(T_n) = \mu^*(T \setminus F)$.

Letting $n \to \infty$ in $(*)$, we get $\mu^*(T) \geq \mu^*(T\cap F) + \mu^*(T \setminus F)$. $\square$

Proof of claim: Let $B_n = T_{n+1} \setminus T_n$. Then $d(B_{n+1}, T_n) \geq \frac{1}{n(n+1)}$.

[If $x \in B_{n+1}$, and $d(x,y) < \frac{1}{n(n+1)}$, then $d(y, F) \leq d(y,x) + d(x,F) < \frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n}$.
So $y \notin T_n$.]



It follows

$$\mu^*(T_{2k+1}) \geq \mu^*(B_{2k} \cup T_{2k-1}) = \mu^*(T_{2k-1}) + \mu^*(B_{2k}) = \cdots = \sum_{j=1}^{k} \mu^*(B_{2j})$$

$$\mu^*(T_{2k}) \geq \mu^*(B_{2k-1} \cup T_{2k-2}) = \cdots = \sum_{j=1}^{k} \mu^*(B_{2j-1})$$

$$\Rightarrow \sum_{j=1}^{\infty} \mu^*(B_{2j}) \leq \lim_{k\to\infty} \mu^*(T_{2k+1}) \leq \lim_{k\to\infty} \mu^*(T) < \infty$$

$$\sum_{j=1}^{\infty} \mu^*(B_{2j-1}) < \infty.$$

$$\Rightarrow \sum_{j=n}^{\infty} \mu^*(B_j) \to 0 \text{ as } n \to +\infty$$

But by def., we have

$$\mu^*(T_n) \leq \mu^*(T\setminus F) = \mu^*\Big(T_n \cup \bigcup_{j=n}^{\infty} B_j\Big) \leq \mu^*(T_n) + \sum_{j=n}^{\infty} \mu^*(B_j).$$

So letting $n \to \infty$, we get

$$\mu^*(T \setminus F) = \lim_{n\to\infty} \mu^*(T_n). \quad \square$$

2. The Hausdorff measure

- Here is another measure defined for subsets in $\mathbb{R}^d$, which is particularly useful for measuring sets that are not "very regular".

Note that $(\mathbb{R}^d, \overset{d(x,y)=}{\|x-y\|})$ is a metric space.

Given any bounded set $A \subset \mathbb{R}^d$, we let $\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$.

Def: For any $\alpha \geq 0$ and any $\delta > 0$, we let

$$h^*_{\alpha,\delta}(A) := \inf\left\{\sum_{k=1}^{\infty} (\operatorname{diam} F_k)^\alpha : A \subset \bigcup_{k=1}^{\infty} F_k, \ \operatorname{diam}(F_k) < \delta\right\}.$$

and let

$$h^*_\alpha(A) = \lim_{\delta \to 0^+} h^*_{\alpha,\delta}(A).$$

The limit exists since $h^*_{\alpha,\delta_1}(A) \leq h^*_{\alpha,\delta_2}(A)$ for $\delta_2 \leq \delta_1$

Prop: h^*_α is a metric outer measure on $\mathbb{R}^d$.

Proof:
- Obviously $h^*_\alpha(\phi) = 0$, $\cdot$ $h^*_\alpha(A_1) \leq h^*_\alpha(A_2)$ for $A_1 \subset A_2$.
- Now let $A_j \subset \mathbb{R}^d$. Choose $F_{j,k}$ s.t. $\operatorname{diam}(F_{j,k}) < \delta$, $A_j \subset \bigcup_{k=1}^{\infty} F_{j,k}$, and

$$\sum_{k=1}^{\infty} (\operatorname{diam} F_{j,k})^\alpha \leq h^*_{\alpha,\delta}(A_j) + \frac{\varepsilon}{2^j}.$$

Then we have (since $\bigcup_{j=1}^{\infty} A_j \subset \bigcup_{j,k} F_{j,k}$)

$$h^*_{\alpha,\delta}\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j,k} (\operatorname{diam} F_{j,k})^\alpha \leq \sum_{j=1}^{\infty} h^*_{\alpha,\delta}(A_j) + \varepsilon.$$

Letting $\delta \to 0$ and $\varepsilon \to 0$, we get $h^*_\alpha\big(\bigcup_{j=1}^{\infty} A_j\big) \leq \sum_{j=1}^{\infty} h^*_\alpha(A_j)$.

- Now suppose $d(A_1, A_2) > 0$. We need to show $h^*_\alpha(A_1 \cup A_2) \geq h^*_\alpha(A_1) + h^*_\alpha(A_2)$.

We take any $\delta < d(A_1, A_2)$. Let $F_1, F_2, \cdots$ be a covering of $A_1 \cup A_2$ with $\operatorname{diam}(F_i) < \delta$. Let $F_j' = F_j \cap A_1$, $F_j'' = F_j \cap A_2$. Then $A_1 \subset \bigcup_j F_j'$, $A_2 \subset \bigcup_j F_j''$. ~~As a consequence~~ On the other hand, $F_j' \cap F_i'' = \phi$ for all i, j.

$$\Rightarrow \sum_j (\operatorname{diam} F_j')^\alpha + \sum_i (\operatorname{diam} F_i'')^\alpha \leq \sum_k (\operatorname{diam} F_k)^\alpha.$$

So we get $h^*_{\alpha,\delta}(A_1) + h^*_{\alpha,\delta}(A_2) \leq h^*_{\alpha,\delta}(A_1 \cup A_2)$

Letting $\delta \to 0$, we get $h^*_\alpha(A_1 \cup A_2) \geq h^*_\alpha(A_1) + h^*_\alpha(A_2)$. $\square$

Cor: h^*_α induces a Borel measure on $\mathbb{R}^d$. The measure is called the Hausdorff measure of dimension α, and is denoted by $\mathcal{H}^\alpha$.

e.g.: $\mathcal{H}^0$ = the counting measure.

· So for each $\alpha \in [0, +\infty)$, one has a measure $\mathcal{H}^\alpha$ on $\mathbb{R}^d$.

The next proposition says that for each given Borel set A, there is essentially only one Hausdorff measure $\mathcal{H}^\alpha$ which is suitable to measure A.

<u>Prop:</u> (1) If $\mathcal{H}^\alpha(A) < \infty$, then for any $\beta > \alpha$, $\mathcal{H}^\beta(A) = 0$.

(2) If $\mathcal{H}^\alpha(A) > 0$, then for any $\beta < \alpha$, $\mathcal{H}^\beta(A) = \infty$.

<u>Proof.</u> Suppose $\alpha < \beta$, and $\operatorname{diam} F < \delta$. Then

$$(\operatorname{diam} F)^\beta = (\operatorname{diam} F)^\alpha \cdot (\operatorname{diam} F)^{\beta-\alpha} \leq \delta^{\beta-\alpha} (\operatorname{diam} F)^\alpha.$$

Applying this to each F_i, where $A \subset \bigcup_{i=1}^\infty F_i$, we get (1).

Same argument gives (2). $\square$

<u>Def:</u> For any Borel set $A \subset \mathbb{R}^d$, the <u>Hausdorff dimension</u> of A is

$$\dim_H(A) := \sup\{\beta : \mathcal{H}^\beta(A) = \infty\} = \inf\{\alpha : \mathcal{H}^\alpha(A) = 0\}$$

<u>Example:</u> Let $\mathcal{C}$ be the Cantor set.

Then $\dim_H \mathcal{C} = \frac{\ln 2}{\ln 3}$.