Real Analysis

Last time

- A set K is compact if any open covering $\{U_\alpha\}$ of K admits a finite subcovering.
  - Closed sets in compact set are compact
  - Any continuous function on a compact set admits max/min value ($\Rightarrow$ It is bounded.)
  - Existence of "partition of unity".
- A Borel measure μ on (X,d) is
  - outer regular if $\forall A \in \mathcal{B}_X$, $\mu(A) = \inf\{\mu(U) : U \supset A,\ U \text{ open}\}$
  - inner regular if $\forall A \in \mathcal{B}_X$, $\mu(A) = \sup\{\mu(K) : K \subset A,\ K \text{ compact}\}$
  - locally finite if $\mu(K) < +\infty$, $\forall$ compact set K

  $\Rightarrow$ $\boxed{\text{Radon measure}}$ (Assuming locally compact and σ-compact)

  [If (X,d) is compact, μ is locally finite, then μ is regular.]

- Riesz representation thm: Suppose (X,d) is compact, $\ell : C(X) \to \mathbb{R}$ is a positive linear functional. Then $\exists$ unique Borel measure, locally finite μ s.t.
$$\ell(f) = \int_X f\, d\mu, \quad \forall f \in C(X).$$

  [(X,d) locally compact and σ-compact, $\ell : C_c(X) \to \mathbb{R}$ positive $\Rightarrow \exists$ unique Radon measure μ.]

Today: L^p-space

1. Spaces and structures

- When we say "space", what do we mean?

  First: A "space" V is a set whose elements are the objects that we want to study.

  Second: These objects are grouped together so that the set admits extra structures.

  Usually the second is more important: It tells us how to choose V
  It tells us how to study V.

- We have learned many "structures":

  (1) linear structure: Given two elements $v_1, v_2 \in V$ and two scalar c_1, c_2,
  $\rightsquigarrow c_1 v_1 + c_2 v_2 \in V$.
  [e.g. $\mathbb{R}^d$, $L^1(\mathbb{R}^d)$, $L^1(X,\mu)$, $C_c(X)$, $\cdots$ Note: $B(0,a)$ is NOT a vector space]

  (2) Multiplication structure: Given two elements $v_1, v_2 \in V$
  $\rightsquigarrow v_1 \cdot v_2 \in V$.
  [e.g. $C_c(X)$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^3$ ($\vec{v}, \vec{w} \to \vec{v} \times \vec{w}$)
  also: "convolution" of functions.   Note: Usually we want the multiplication structure to be compatible with the linear structure.]

(3) Norm structure. Norm is a structure on a vector space, $\|\cdot\|: V \to \mathbb{R}$ s.t.

(1) $\|0\| = 0$, and $\|v\| > 0$ for $\forall v \neq 0$.

(2) $\|\alpha v\| = |\alpha| \cdot \|v\|$ for any $v \in V$ and any scalar α.

(3) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ (triangle inequality)

[e.g. $\mathbb{R}^d$, $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$; $\mathbb{R}^d$, $\|x\| = |x_1| + \cdots + |x_d|$;
$\mathbb{R}^d$, $\|x\| = \max(|x_1|, \cdots, |x_d|)$, $L^1(\mathbb{R}^d)$, $\|f\| = \int_{\mathbb{R}^d} |f(x)|\,dx$.]

(4) Metric structure. Metric d is a function $d: X \times X \to \mathbb{R}$ s.t.

(1) $d(x, x) = 0$; $d(x, y) > 0$, $\forall x \neq y$

(2) $d(x, y) = d(y, x)$.

(3) $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequalities)

Note: Any normed vector space is a metric space, since one can define

$$d(v_1, v_2) = \|v_1 - v_2\|$$

But metric spaces need NOT be vector spaces.

[e.g. $[0,1]$, $d(x,y) = |x-y|$; $[0,1]$, $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$, ...]

(5) Topological structure: open sets $\longrightarrow$ closed sets, compact sets
$\longrightarrow$ convergence.
$\longrightarrow$ continuous function.
$\longrightarrow$ connectedness, separatedness

Note: • For metric space, one usually use the topology generated by open balls.
• But that is NOT the only topology.
• For spaces of functions, it is important to know how a sequence of functions converges to a function (different modes)

(6) Inner product structure (Angle) It is also a structure on vector spaces

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$

that satisfies (1) $\langle v, w \rangle = \langle w, v \rangle$

(2) $\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$.

(3) $\langle 0, 0 \rangle = 0$, $\langle v, v \rangle > 0$ for $v \neq 0$.

Note: Any inner product induces a norm by setting

$$\|v\| = \langle v, v \rangle.$$

• One can define the conception of "angle" between elements in an inner product space by

$$\angle(v, w) := \arccos \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}.$$

$\longrightarrow$ orthogonality $v \perp w \Leftrightarrow \langle v, w \rangle = 0$.

(7). Order structure: $\rightsquigarrow$ how to compare elements.

For a general set, it is hard to get an order structure.

But $\mathbb{R}$ has a nice order structure

$\rightsquigarrow$ "$f \geq 0$", "$f \geq g$", $\sup\{f_1, f_2\}$, ---

Note: This is one of the reasons that real analysis breaks down over other number fields.

(8). Symmetry structure: "group" that preserves some given structures.

e.g.: · On $\mathbb{R}^d$, one have translation: $x \rightsquigarrow x - x_0$.

one has $GL(d, \mathbb{R})$: $x \rightsquigarrow Ax$, $A \in GL(d, \mathbb{R})$.

· For a measurable space (X, μ): $\{\phi: X \to X;\ \phi^* \mu = \mu\}$ ← measure preserving group.

(9). Measure structure: Of course measure is also an extra structure using which one can "find volume" or "integrate functions"

(10). Duality. For a given space V, sometimes one would like to find another space V^* whose elements can be "paired" with V,

i.e. $(l, v) \longmapsto \langle\langle l, v \rangle\rangle \in \mathbb{R}$ (or $\mathbb{C}$, --).

e.g.: · If V is inner product space, V can be paired with V by

$\langle\langle v, w \rangle\rangle := \langle v, w \rangle$

· V^* be "linear functionals" on a vector space

$\langle\langle l, v \rangle\rangle := l(v)$.

· One can pair "space of functions" with "space of measures"

$\langle\langle \mu, f \rangle\rangle := \int_X f \, d\mu$.

· One can pair X with "space of functions on X"

$\langle\langle x, f \rangle\rangle := f(x)$.

[This is just the δ-measure at x.]

Note: · Riesz representation thm $\rightsquigarrow$ The dual space of $C_c(X)$

$\simeq$ The space of Radon measures.

· "Dual space" plays a crucial role in "functional analysis"

2. L^p-spaces

- Let $X \subset \mathbb{R}^d$ be a measurable subset, or more generally, let $(X, \mathcal{F}, \mu)$ be a measure space.

Def (1) Given any measurable function f on X, we define the L^p-norm of f to be

$$\|f\|_{L^p} := \left(\int_X |f|^p \, d\mu \right)^{1/p} \qquad (0 < p < +\infty)$$

(2) The L^p-space $L^p(X, \mathcal{F}, \mu)$ (usually we write $L^p(X)$, $L^p(X, \mu)$)

$$L^p(X) := \{ f : f \text{ is measurable on } X, \text{ and } \|f\|_{L^p} < +\infty \}.$$

Example: $L^p(\mathbb{R}^d)$ (Lecture 11)

Example: Let $X = \mathbb{N}$, $\mathcal{F} = P(X)$, $\mu =$ the counting measure $\#$.

Then a function on X is just a sequence $(a_1, a_2, \cdots\cdots) =: a$.

- Any function is measurable since $\mathcal{F} = P(X)$.
- The L^p-norm of $a = (a_1, a_2, \cdots)$ is

$$\|a\|_{L^p} = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}$$

So the L^p-space is

$$\ell^p := \{ (a_1, a_2, \cdots) : \sum_{i=1}^{\infty} |a_i|^p < +\infty \}.$$

Rmk: When we talk about "the space $L^p(X)$", we will regard two a.e. equal functions as "the same element" in $L^p(X)$. In other words, $L^p(X)$ is really a quotient

$$L^p(X) = \{ f : f \text{ is measurable}, \|f\|_{L^p} < +\infty \} / \sim$$

w.r.t. the equivalence relation $f_1 \sim f_2 \Leftrightarrow f_1 = f_2$ a.e.

- The first structure on $L^p(X)$ is the linear structure:
- **Prop.** For any $0 < p < +\infty$, $L^p(X)$ is a vector space.

Proof. Obviously if $\|f\|_{L^p} < +\infty$, then for any scalar a,

$$\|af\|_{L^p} = \left(\int_X |a|^p |f|^p \, d\mu \right)^{1/p} = |a| \cdot \|f\|_{L^p} < +\infty.$$

It remains to show $f_1, f_2 \in L^p \Rightarrow f_1 + f_2 \in L^p$. This follows from the pointwise inequality

$$|f_1(x) + f_2(x)|^p \le \left(2 \max(|f_1(x)|, |f_2(x)|) \right)^p$$
$$\le 2^p \left(|f_1(x)|^p + |f_2(x)|^p \right). \qquad \square$$

• Next we will show that $\|\cdot\|_{L^p}$ is a norm on $L^p(X)$, (but this is true only for $p \geq 1$).

<u>Thm</u>: For any $p \geq 1$, $\|\cdot\|_{L^p}$ is a norm on $L^p(X)$. In other words,

(1) $\|f\|_{L^p} = 0 \iff f = 0$.

(2) $\|cf\|_{L^p} = |c| \cdot \|f\|_{L^p}$

(3) $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.

<u>Note</u>: • (1), (2) hold for all $p > 0$.

• (3) is known as the <u>Minkowski inequality</u>, which we will prove below.

• For $0 < p < 1$, (3) fails, and in fact one has ~~the reverse Minkowski inequality~~

$$\cancel{\quad} \quad \|f\|_{L^p} + \|g\|_{L^p} \underset{\text{the reverse Minkowski inequality}}{\leq} \| |f| + |g| \|_{L^p} \leq 2^{\frac{1}{p}-1}\left(\|f\|_{L^p} + \|g\|_{L^p}\right).$$

• To prove the Minkowski inequality (3), we first prove

<u>Thm</u>. (Hölder's inequality) Suppose $p, q, r > 0$ be s.t. $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.
Suppose $f \in L^p$, $g \in L^q$.
Then $fg \in L^r$, and $\|fg\|_{L^r} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$

<u>Proof</u>: WLOG, we may assume $f, g \geq 0$. [Otherwise use $|f|$, $|g|$]

WLOG, we may assume $A := \|f\|_{L^p} > 0$, $B := \|g\|_{L^q} > 0$. [Otherwise $f = 0$ a.e. or $g = 0$ a.e. $\Rightarrow$ both sides are 0.]

• ~~We will use~~ First assume $r = 1$. We will need

<u>Young's Inequality</u>: $a, b \geq 0 \Rightarrow ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$.

Then
$$\frac{\|fg\|_{L^1}}{\|f\|_{L^p} \cdot \|g\|_{L^q}} = \int_X \frac{f}{A} \cdot \frac{g}{B}\, d\mu \leq \int_X \left(\frac{1}{p}\left(\frac{f}{A}\right)^p + \frac{1}{q}\left(\frac{g}{B}\right)^q\right) d\mu$$
$$= \frac{1}{p} \cdot \frac{1}{A^p}\int_X f^p\, d\mu + \frac{1}{q} \cdot \frac{1}{B^q}\int_X g^q\, d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

i.e. $\|fg\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$ if $\frac{1}{p} + \frac{1}{q} = 1$.

• <u>For general r</u>, we notice that for $s > 0$, $\|f^s\|_{L^p} = \|f\|_{L^{ps}}^s$. So

$$\|fg\|_{L^r} = \|(fg)^{r \cdot \frac{1}{r}}\|_{L^r} = \|(fg)^r\|_{L^1}^{1/r} \leq \|f^r\|_{L^{p/r}}^{\frac{1}{r}} \cdot \|g^r\|_{L^{q/r}}^{1/r} = \|f\|_{L^p} \cdot \|g\|_{L^q}. \quad \square$$

Now we are ready to prove

<u>Thm</u>. (Minkowski inequality) Suppose $p \geq 1$. Then $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.

<u>Proof</u>:
$$\|f+g\|_{L^p}^p = \int_X f(f+g)^{p-1}\, d\mu + \int_X g(f+g)^{p-1}\, d\mu$$
$$\leq \|f\|_{L^p} \cdot \|(f+g)^{p-1}\|_{L^q} + \|g\|_{L^p} \cdot \|(f+g)^{p-1}\|_{L^q}$$
$$= \left(\|f\|_{L^p} + \|g\|_{L^p}\right) \|f+g\|_{L^{(p-1)q}}^{p-1} = \left(\|f\|_{L^p} + \|g\|_{L^p}\right)\|f+g\|_{L^p}^{p-1}. \quad \square$$

Let q be s.t. $\frac{1}{p} + \frac{1}{q} = 1$
Then $pq - q > p$.