Real Analysis 3

Last time

- **Structures**: ("Algebraic, operation") Linear, multiplication, symmetry, duality
  ("Topological/geometric, position") Topology, metric, norm, inner product
  ("Analytic, limit") Measure, smooth,
  ("order, relation") order,

- **$L^p(X,\mu)$**: (For $0<p<+\infty$) Vector space structure
  (For $1\le p<+\infty$) Normed vector space. $\|f\|_{L^p}=\left(\int_X |f|^p\,d\mu\right)^{1/p}$
  - Hölder's inequality $\|fg\|_{L^r}\le \|f\|_{L^p}\cdot\|g\|_{L^q}$ for $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. $(0<p,q,r<+\infty)$
  - Minkowski inequality. $\|f+g\|_{L^p}\le\|f\|_{L^p}+\|g\|_{L^p}$ $(p\ge 1)$

Note: Minkowski inequality
$\iff$ Triangle inequality of $\|\cdot\|$
$\iff$ The norm function is convex
$\|\lambda f+(1-\lambda)g\|_{L^p}\le\lambda\|f\|_{L^p}+(1-\lambda)\|g\|_{L^p}$

Today. More on L^p-spaces

1. L^∞-space.

- We have defined the spaces $L^p(X)$ for all $1\le p<\infty$.
  Note that one function could belong to different $L^p(X)$.
  In last problem set, we have seen that for $\mu(X)<\infty$,
  - $p_1<p_2 \Rightarrow L^{p_2}(X)\subset L^{p_1}(X)$.
  - $\|f\|_{L^p}\to\|f\|_{L^{p_0}}$ for $p\to p_0^-$.

Note. We don't have $\|f\|_{L^p}\to\|f\|_{L^{p_0}}$ for $p\to p_0^+$.
e.g. Let $X=[0,\frac{1}{2}]$, $f(x)=\frac{1}{x(\log x)^2}$.
Then by monotone convergence,
$\int_X |f|\,dx=(\log 2)^{-1}<\infty$.
But for any $p>1$,
$\int_X |f|^p\,dx=+\infty$.

- A natural question is: what happens for $\|f\|_{L^p}$ as $p\to+\infty$?
  Let's start with a simple example.
  $X=[0,4]$, $d\mu=$ Lebesgue.
  $$f(x)=\begin{cases}4, & x=0\\ 3, & 0<x\le 1\\ 2, & 1<x\le 2\\ 1, & 2<x\le 4\end{cases}$$
  Then $\|f\|_{L^p(X)}=\left(\int_0^4|f|^p\,dx\right)^{1/p}=(3^p+2^p+2\cdot 1^p)^{1/p}\longrightarrow 3$ as $p\to+\infty$.

This is the upper bound of $|f|$ if we ignore the value 4 of f which is only attained by f on a measure zero set.
According to Lebesgue, we should ignore events that only happens on a measure zero set.

By computing more examples, you can easily find by yourself that $\|f\|_{L^p}$ should approximates the upper bound of $|f|$ (ignoring measure zero sets).

- **Def.** Let f be any measurable function on $(X, \mathcal{F}, \mu)$.
  (1) We say f is <u>essentially bounded</u> if $\exists M$ s.t. $|f(x)| \le M$ for a.e. $x \in X$.
  (2) The <u>L^∞-norm</u> of f is $\|f\|_{L^\infty(X)} := \inf\{M : |f(x)| \le M \text{ for a.e. } x \in X\}$.
  (3) The <u>L^∞-space</u> $L^\infty(X) = \{f : f \text{ is measurable and } \|f\|_{L^\infty} < +\infty\}$.

As for $p<\infty$, one has

Thm. $L^\infty(X)$ is a vector space, and $\|\cdot\|_{L^\infty}$ is a norm on $L^\infty(X)$.

Proof.
- $f \in L^\infty(X)$, $a \in \mathbb{R} \Rightarrow \|af\|_{L^\infty} = a\|f\|_{L^\infty} < \infty$.
- $f, g \in L^\infty(X) \Rightarrow |f|+|g| \le \|f\|_{L^\infty} + \|g\|_{L^\infty}$, a.e. x $\Rightarrow \|f+g\|_{L^\infty} \le \|f\|_{L^\infty} + \|g\|_{L^\infty}$.

$\Rightarrow L^\infty(X)$ is a vector space

- $\|0\|_{L^\infty} = 0$, $\|f\|_{L^\infty} = 0 \Leftrightarrow f = 0$ a.e. ✓
- $\|af\|_{L^\infty} = a\|f\|_{L^\infty}$ ✓
- $\|f+g\|_{L^\infty} \le \|f\|_{L^\infty} + \|g\|_{L^\infty}$ ✓

$\Rightarrow \|\cdot\|_{L^\infty}$ is a norm. □

Note. We have already proved the <u>Minkowski inequality</u> for $L^\infty(X)$. So
$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}, \qquad 1 \le p \le +\infty$$

Similarly, we can also prove <u>Hölder's inequality</u>
~~Thm (Hölder's equality)~~ $\|fg\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$, $0 < p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

[The only thing left is to check the inequality for $q = r$, $p = +\infty$. or <u>$p = r$, $q = +\infty$</u> by symmetry.
But we do have $|fg| \le \|f\|_{L^\infty}|g|$, a.e.
$\Rightarrow \|fg\|_{L^r} \le \|f\|_{L^\infty} \cdot \|g\|_{L^r}$. Done.]

Rmk. For $X = \mathbb{N}$, $\mathcal{F} = P(X)$, and $\mu = \#$, we usually denote $L^\infty(X)$ by ℓ^∞.
Since each point has positive measure $(\#\{n\} = 1)$,
$$\|(a_1, a_2, \cdots)\|_{\ell^\infty} = \sup_{k \ge 1} |a_k|.$$
Thus $\ell^\infty = \{(a_1, a_2, \cdots) : \exists M \text{ s.t. } |a_k| \le M, \forall k\}$.

Rmk. One can easily write down the Minkowski inequality and Hölder's inequality involving more than two functions:
- $\|f_1 + f_2 + \cdots + f_n\|_{L^p} \le \|f_1\|_{L^p} + \cdots + \|f_n\|_{L^p}$, $1 \le p \le \infty$
- $\|f_1 \cdot f_2 \cdot \cdots \cdot f_n\|_{L^r} \le \|f_1\|_{L^{p_1}} \cdot \cdots \cdot \|f_n\|_{L^{p_n}}$, $0 < p_i \le +\infty$, $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_n}$.

$\boxed{\text{Try to prove this}}$

2. L^p-spaces as metric spaces

- For $1 \le p \le +\infty$, the norm $\|\cdot\|_{L^p}$ on $L^p(X)$ induces a metric on $L^p(X)$:

$$d_{L^p}(f,g) := \|f-g\|_{L^p}$$

One can easily check ① $d_{L^p}(f,g) \ge 0$, and "=" $\Leftrightarrow f=g$ (a.e.)

② $d_{L^p}(f,g) = d_{L^p}(g,f)$

③ $d_{L^p}(f,h) \le d_{L^p}(f,g) + d_{L^p}(g,h)$.

- We say a sequence $f_n \in L^p(X)$ <u>converges in L^p-norm</u> to $f \in L^p(X)$ if $\|f_n - f\|_{L^p} \to 0$ as $n \to \infty$.

We say a sequence $f_n \in L^p(X)$ is a <u>Cauchy sequence</u> in $L^p(X)$ if $\forall \varepsilon > 0, \exists N > 0$ s.t.

$$\|f_i - f_j\|_{L^p} < \varepsilon, \quad \forall i,j \ge N.$$

A metric space is <u>complete</u> if any Cauchy sequence converges.

A normed vector space is called a <u>Banach space</u> if it is complete.

- <u>Thm.</u> For any $1 \le p \le \infty$, the space $L^p(X)$ is a Banach space. [Riesz-Fischer]

<u>Proof</u>: [The proof is similar to the proof of completeness of $L^1(A)$ in Lecture 12]

Let $\{f_n\}$ be a Cauchy sequence in $L^p(X)$.

$\boxed{\text{If } 1 \le p < \infty}$ <u>Step 1</u>: Pick a subsequence f_{n_k} s.t. $\|f_{n_k} - f_{n_{k-1}}\|_{L^p(X)} < \frac{1}{2^k}$ by condition.

<u>Step 2</u>: Construct $g \in L^p(X)$ s.t. g dominates f_{n_k}

Let $g_k = |f_{n_1}| + \sum_{i=2}^{k} |f_{n_i} - f_{n_{i-1}}| \longrightarrow g$ by monotone convergence.

<u>Note</u>: • $\|g_k\|_{L^p} \le \|f_{n_1}\|_{L^p} + \sum_{i=2}^{k} \frac{1}{2^i} < \|f_{n_1}\|_{L^p} + 1 \Rightarrow g \in L^p(X)$.

• $|f_{n_k}| = |f_{n_1} + (f_{n_2} - f_{n_1}) + \dots + (f_{n_k} - f_{n_{k-1}})| \Rightarrow |f_{n_k}| \le g, \forall k$.

<u>Step 3</u>: Let $f = f_{n_1} + \sum_{i=2}^{\infty} (f_{n_i} - f_{n_{i-1}})$. Then $|f| \le g \Rightarrow f \in L^p(X)$.

Since $f_{n_k} = f_{n_1} + \sum_{i=2}^{k} (f_{n_i} - f_{n_{i-1}}) \to f$, we apply dominate convergence to the sequence $|f_{n_k} - f|^p \le 2^p |f|^p$

to get $\|f_{n_k} - f\|_{L^p} \to 0$.

<u>Step 4</u>: Use triangle inequality to prove $\|f_k - f\|_{L^p} \to 0$.

$\boxed{\text{If } p = \infty}$ <u>Step 1</u>: Same.

<u>Step 2</u>: Let $f = f_{n_1} + \sum_{i=2}^{\infty} (f_{n_i} - f_{n_{i-1}})$. Then by def., $\|f\|_{L^\infty} \le \|f_{n_1}\|_{L^\infty} + 1 < \infty$.

and $\|f_{n_k} - f\|_{L^\infty} \le \frac{1}{2^{k-1}} \to 0$. So $f_{n_k} \to f$ in $L^\infty(X)$.

<u>Step 3</u>: ~~Therey~~ Triangle inequality. $\square$

• **Def.** We say a metric space $\mathbb{X}$ is <u>separable</u> if it contains a countable dense subset. In other words, there is a countable set $\mathbb{X}_0=\{x_1,x_2,x_3,\dots\}\subset\mathbb{X}$ s.t. $\overline{\mathbb{X}_0}=\mathbb{X}$.

[Note: $\overline{\mathbb{X}_0}=\{y\in\mathbb{X}: \exists x_{n_k}\in\mathbb{X}_0$ s.t. $x_{n_k}\to y$ as $k\to\infty\}$.]

e.g. • $\mathbb{R}^d$ is separable since $\mathbb{Q}^d\subset\mathbb{R}^d$ is a countable dense subset.

• $L^\infty([0,1])$ is NOT separable (exercise)

Thm. Suppose ~~μ is a σ-finite measure on X (i.e. $X=\bigcup_{k=1}^\infty X_k$, and $\mu(X_k)<\infty$, $\forall k$)~~, and ~~X is separable~~ there is a countable family of open sets $U_1,U_2,\dots$ of X s.t. ① $\mu(U_i)<+\infty$, $\forall i$. ② Any open set in X is a union of some open sets U_i's.

Then for $1\le p<\infty$, $L^p(X)$ is separable.

This is true for $\mathbb{R}^d$, since we can take U_i to be open balls of the form $B(a_i,r_i)$, $a_i\in\mathbb{Q}^d$, $r_i\in\mathbb{Q}$

We first prove a Lemma. ~~Assume μ is σ-finite~~

Lemma. For $0<p<\infty$, the set
$$S:=\left\{f=\sum_{i=1}^n a_i\chi_{A_i} : \mu(A_i)<\infty\right\}$$
is dense in $L^p(X)$.

Proof: Obviously $S\subset L^p(X)$. We need to prove S is dense.

Observation 1: $B:=\{f\in L^p : \exists M \text{ s.t. } |f|\le M\}$ is dense in $L^p(X)$.

Reason: $\forall f\in L^p(X)$, let $f_M:=\begin{cases} f(x), & \text{if } |f(x)|\le M\\ M, & \text{if } |f(x)|>M.\end{cases}$

Then $|f_M|\le f \Rightarrow f_M\in L^p$.

Since $|f_M|\nearrow f$ as $M\nearrow\infty$, the monotone convergence thm implies that $f_M\to f$ in L^p.

Observation 2: $BB:=\{f\in B: \exists N \text{ s.t. } \mu(\operatorname{supp} f)<N\}$ is dense in $L^p(X)$

Reason: Suppose $f\in L^p(X)$, and $f_n\in B$ s.t. $f_n\to f$ in L^p.

Note: $\int_X |f|^p\,d\mu=:A<\infty \Rightarrow \mu(\{x: |f|>\frac{1}{m}\})\le A m^p$.

Let $f_{n,m}=\begin{cases} f_n(x), & |f_n(x)|>\frac{1}{m}\\ 0, & |f_n(x)|<\frac{1}{m}.\end{cases}$

Then $f_{n,m}\in BB$, and $f_{n,m}\nearrow f_n$ as $m\to\infty$.

By monotone convergence, $f_{n,m}\to f_n$ in L^p.

The diagonal trick implies $f_{n,m(n)}\to f$ in L^p.

Observation 3: For any $f\in BB$, $\exists f_k\in S$ s.t. $f_k\to f$ in L^p.

Reason: Let $A_i=\{x: \frac{i}{2^k}<f(x)\le\frac{i+1}{2^k}\}$, $i=-\infty,\dots,+\infty$.

Then only finitely many $A_i\ne\emptyset$, and each $\mu(A_i)\le\mu(\operatorname{supp} f)<\infty$.

Moreover, $f_i:=\sum_i \frac{i}{2^k}\chi_{A_i}\to f$ in L^p.

Finally apply the ~~diagonal trick~~ Triangle inequality. □

Now we prove the theorem that $L^p(X)$ is separable.

<u>Proof</u>: It is enough to prove that there exists a countable set

$$V_0 \subset L^p(X)$$

s.t. any $f \in \mathcal{S}$ can be approximated by functions in V_0 under the L^p-norm.

We take $V_0 = \left\{ \sum_{I} r_I \chi_{U_I} : r_I \in \mathbb{Q}, \cancel{k \in \mathbb{N}} \right\}$, $I = (i_1, \cdots, i_k)$, $|i_j| < N$, $U_I = U_{i_1} \cup \cdots \cup U_{i_k}$.

Then V_0 is a countable set.

Let $f = \sum_{i=1}^{n} a_i \chi_{A_i} \in \mathcal{S}$. It is enough to prove $a_i \chi_{A_i}$ can be approximated by elements in V_0.

- - - - - - - - - - χ_{A_i} - - - - - - - - -

(since one can choose $r_n \in \mathbb{Q}$ s.t. $r_n \to a_i$)

Finally to approximate χ_{A_i}, where $\mu(A_i) < \infty$, we notice that $\exists$ open set $U \supset A_i$ s.t. $\mu(U) < \mu(A_i) + \varepsilon$.

Also $\exists$ countably many $U_{i_1}, U_{i_2}, \cdots$ s.t. $U = \bigcup_{j=1}^{\infty} U_{i_j}$.

By monotone convergence, $\exists K$ s.t. $\mu\left(U \setminus \bigcup_{j=1}^{K} U_{i_j}\right) < \varepsilon$.

Now let $\varphi = \chi_{\bigcup_{j=1}^{K} U_{i_j}}$. Then $\varphi \in V_0$, and $\|\varphi - \chi_U\| < \varepsilon$.

So the proof is complete. $\square$