Real Analysis

Lecture 22
05/17/2018

Last time.

- L^∞-space: $\|f\|_{L^\infty} = \inf\{M : |f(x)| \le M, \text{ a.e. } x\}$
  - $\|f_1 + f_2 + \cdots + f_n\|_{L^p} \le \|f_1\|_{L^p} + \cdots + \|f_n\|_{L^p}$, $1 \le p \le \infty$
  - $\|f_1 \cdot f_2 \cdot \cdots \cdot f_n\|_{L^r} \le \|f_1\|_{L^{p_1}} \cdot \cdots \cdot \|f_n\|_{L^{p_n}}$, $0 < p_i, r \le \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}$.
- Metric structure
  - $1 \le p \le \infty$: $L^p(X)$ is a Banach space (i.e. "complete").
  - $1 \le p < \infty$: $L^p(\mathbb{R}^d)$ is separable. (i.e. contains a countable dense subset.)
    - $0 < p < \infty$: $\mathcal{S} = \{\varphi = \sum_{i=1}^{k} a_i \chi_{A_i} : \mu(A_i) < \infty, k \in \mathbb{N}, a_i \in \mathbb{R}\}$ is dense in $L^p(X)$.

Today. $L^2(X)$: The inner product structure

1. $L^2(X)$.

- Now consider the space $L^2(X)$. We have seen
  - $\|f_1 + f_2\|_{L^2} \le \|f_1\|_{L^2} + \|f_2\|_{L^2}$.
  - $\|f_1 f_2\|_{L^1} \le \|f_1\|_{L^2} \cdot \|f_2\|_{L^2}$ $\leftarrow$ Cauchy-Schwartz inequality
  - $L^2(X)$ is complete.
  - $L^2(X)$ is separable if $X = \mathbb{R}^d$ or "......"
- Among all $L^p(X)$, the space $L^2(X)$ is special, since it admits an "inner product" structure.

Def. The inner product between $f, g \in L^2(X)$ is
$$\langle f, g \rangle := \int_X fg \, d\mu.$$

Rmk. (1) $f, g \in L^2(X) \Rightarrow f \cdot g \in L^1(X)$. ("well-defined")
(2) If f, g are complex-valued, then one define $\langle f, g \rangle = \int_X f \bar{g} \, d\mu$.

Prop. The inner product $\langle \cdot, \cdot \rangle : L^2(X) \times L^2(X) \to \mathbb{R}$ satisfies
(1) $\langle a_1 f_1 + a_2 f_2, g \rangle = a_1 \langle f_1, g \rangle + a_2 \langle f_2, g \rangle$
(2) $\langle f, g \rangle = \langle g, f \rangle$
(3) $\langle f, f \rangle \ge 0$, and $\langle f, f \rangle = 0 \Leftrightarrow f = 0$.

Proof. Trivial. $\square$

Note. The Cauchy-Schwartz inequality becomes
$$|\langle f, g \rangle| \le \|f\|_{L^2} \cdot \|g\|_{L^2}$$

· **Rmk**: · In general, given any vector space V, an inner product structure on V is a map $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies (1), (2), (3).

· Given any inner product space $(V, \langle \cdot,\cdot \rangle)$, ~~we call V a Hilbert space~~ one can define a norm on V by

$$\|v\| := \sqrt{\langle v, v\rangle}.$$

(and thus a distance $d(v,w) = \|v-w\|$)

· An inner product vector space V is called a Hilbert space if the induced metric structure is complete. [So: Hilbert space must be Banach space.]

· Example: $\mathbb{R}^d$ is a Hilbert space, where $\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{d} x_i y_i$

$\boxed{\mathbb{R}^d = L^2(X) \text{ where } X = \{1,2,\cdots,d\},\ \mu = \#}$

Example: $L^2(X)$ is a Hilbert space.

Example: $C([0,1])$ is an inner product space, but not a Hilbert space

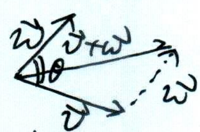
$$\left(\langle f, g\rangle = \int_0^1 f(x) g(x)\, dx\right)$$

· Now let V be any inner product space, $v, w \in V$. Then

$$\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle$$
$$= \|v\|^2 + 2\langle v,w\rangle + \|w\|^2.$$

Compare this with the cosine law of planar triangle

$$\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$$



It is natural to define the angle θ between $v, w \in V$ to be

$$\theta = \arccos \frac{\langle v,w\rangle}{\|\vec{v}\|\cdot\|\vec{w}\|}.$$

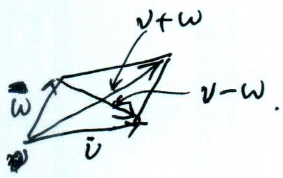
In particular, we say $v \perp w$ if $\theta = \frac{\pi}{2}$, i.e. $\langle v,w\rangle = 0$.

· **Prop**: (parallelogram law) Let V be any inner product space. We have

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

Proof: $\|v \pm w\|^2 = \|v\|^2 + \|w\|^2 \pm 2\langle v,w\rangle.$

$$\Rightarrow \|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2. \quad \square$$



One can check that for any $p \neq 2$, $\exists f, g \in L^p(X)$ s.t.

$$\|f+g\|_{L^p}^2 + \|f-g\|_{L^p}^2 \neq 2\|f\|_{L^p}^2 + 2\|g\|_{L^p}^2.$$

In other words, one can't find an inner product structure on $L^p(X)$ $(p\neq 2)$ s.t. the induced norm is $\|\cdot\|_{L^p(X)}$.

2. The orthogonal projection

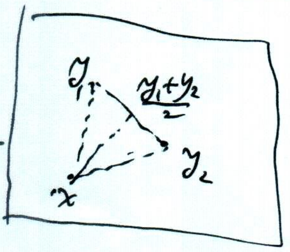
- So a Hilbert space H is – an inner product space ($\Leftrightarrow$ parallelogram law)
  – complete ($\Leftrightarrow$ Any Cauchy sequence converges.)

Using these two properties, we can prove

Thm (Existence of minimizers) Let H be a Hilbert space, $K \subset H$ a non-empty closed convex subset. Let $x \in H$ be any element in H. Then $\exists$ unique $y \in K$ s.t.
$$\|x-y\| = \inf\{\|x-z\| : z \in K\}$$

K is convex $\Leftrightarrow$ $\forall y, z \in K$, $0<t<1$, one has $ty+(1-t)z \in K$

Proof: (Uniqueness) Suppose $y_1, y_2 \in K$, $y_1 \neq y_2$, and
$$\|x-y_1\| = \|x-y_2\| = \inf\{\|x-z\| : z \in K\}.$$
Then by parallelogram law,
$$\|x-y_1\|^2 + \|x-y_2\|^2 = 2\left(\left\|x-\frac{y_1+y_2}{2}\right\|^2 + \left\|\frac{y_1-y_2}{2}\right\|^2\right)$$
$$\Rightarrow \left\|x-\frac{y_1+y_2}{2}\right\|^2 < \frac{1}{2}\left(\|x-y_1\|^2 + \|x-y_2\|^2\right)$$
$$\Rightarrow \left\|x-\frac{y_1+y_2}{2}\right\| < \inf\{\|x-z\| : z \in K\}.$$



But $\frac{y_1+y_2}{2} \in K$ (by convexity), so we get a contradiction.

(Existence). Take $z_n \in K$ s.t. $\|x-z_n\| \to d = \inf\{\|x-z\| : z \in K\} < \infty$.

By convexity, $\frac{z_n+z_m}{2} \in K$. Then
$$d^2 \le \underbrace{\left\|x-\frac{z_n+z_m}{2}\right\|^2}_{A} \le \underbrace{\frac{1}{2}\left(\|x-z_n\|^2 + \|x-z_m\|^2\right)}_{B} \searrow d^2.$$
So we conclude that
$$\|z_n - z_m\|^2 \le 4(B-A) \to 0 \quad \text{as } n, m \to +\infty.$$
As a consequence, $\{z_n\}$ is a Cauchy sequence and thus $\exists y \in H$ s.t. $z_n \to y$ as $n \to \infty$.

Since K is closed, $z_n \in K$, we get $y \in K$.

Finally it is easy to see that $\|x-z_n\| \to \|x-y\|$. So $\|x-y\| = d$. □

- **Rmk**. This property holds for "uniformly convex Banach spaces" including $L^p(X)$ $(1<p<\infty)$. But it fails for $L^1(X)$ and $L^\infty(X)$.

- Since every vector subspace is convex, we immediately get

<u>Cor.</u>: Let H be a Hilbert space, and $V \subset H$ a closed subspace. Then for any $x \in H$, there exists a unique $x_V \in V$ and $x_{V^\perp} \in V^\perp = \{w : \langle v, w\rangle = 0,\ \forall v \in V\}$ s.t. $x = x_V + x_{V^\perp}$. Moreover, $\|x - x_V\| = \inf\{\|x - z\| : z \in V\}$.

<u>Proof.</u> Let x_V be s.t. $\|x - x_V\| = \inf\{\|x - z\| : z \in V\}$, and let $x_{V^\perp} = x - x_V$.

We only need to show $x_{V^\perp} \in V^\perp$.

~~In fact, take any $v \in V$, we have $\langle x_{V^\perp}, v\rangle = \langle x - x_V, v\rangle$~~

Suppose $\exists\, v \in V$ s.t. $\langle x_{V^\perp}, v\rangle \neq 0 \longrightarrow$ Can pick $\mathring{v} \in V$ s.t. $\langle x_{V^\perp}, v\rangle = 1$.

For ε small, we have

$$\|x - x_V - \varepsilon v\|^2 = \|x_{V^\perp} - \varepsilon v\|^2 = \|x_{V^\perp}\|^2 - 2\varepsilon\langle x_{V^\perp}, v\rangle + \varepsilon^2\|v\|^2$$

$$= \|x_{V^\perp}\|^2 - 2\varepsilon + \varepsilon^2\|v\|^2 < \|x_{V^\perp}\|^2 = \|x - x_V\|^2.$$

But $x_V + \varepsilon v \in V$. This contradicts with the fact

$$\|x - x_V\| = \inf\{\|x - z\| : z \in V\}.$$

$\square$

<u>Rmk.</u>
- We thus get an "orthogonal projection" $\pi_V : H \to V$, $x \mapsto x_V$.
- One can prove: $V^\perp \subset H$ is also a closed vector subspace, and $H = V \oplus V^\perp$.

- <u>Recall.</u>
  - Any linear function on $\mathbb{R}^d$ is of the form

$$l_a : \mathbb{R}^d \to \mathbb{R},\quad (x_1, x_2, \dots, x_d) \mapsto a_1x_1 + a_2x_2 + \dots + a_dx_d \ (= \langle a, x\rangle)$$

<u>Note</u>: $a = (a_1, \dots, a_d) \in \mathbb{R}^d$.

So any linear function l_a on $\mathbb{R}^d \longleftrightarrow$ a point $a \in \mathbb{R}^d$.

- Now let H be any Hilbert space.

Then for any $v \in H$, one can define a continuous linear functional

$$L_v : H \to \mathbb{R},\quad L_v(w) = \langle v, w\rangle.$$

L_v is continuous since we have the Cauchy-Schwartz inequality

It turns out that these are all possible ~~bounded~~ continuous linear functionals on H!

<u>Thm.</u> (Riesz representation thm for Hilbert spaces)

Let H be a Hilbert space, and $L : H \to \mathbb{R}$ be a continuous linear functional. Then $\exists$ unique $v \in H$ s.t. $L = L_v$.

Proof: (Uniqueness) Suppose $L_{v_1} = L = L_{v_2}$. Then

$$L_{v_1}(v_1 - v_2) = L_{v_2}(v_1 - v_2).$$

i.e. $\langle v_1, v_1 - v_2 \rangle = \langle v_2, v_1 - v_2 \rangle$

$\Rightarrow \langle v_1 - v_2, v_1 - v_2 \rangle = 0$

$\Rightarrow v_1 = v_2$.

(Existence) WLOG, suppose L is not identically zero.

Let $V = \{x \in H : L(x) = 0\} = L^{-1}(0) = \ker(L)$

Since L is continuous, V is closed subspace of H, and $V \subsetneq H$.

As a consequence, $V^\perp \neq 0$. [Reason: Take any $x \in H$ but $x \notin V$. Then $x_{V^\perp} = x - x_V \neq 0$.]

Pick $w \in V^\perp$ s.t. $\|w\| = 1$. [Reason: $V^\perp$ is a vector space.]

Then $L(w) \neq 0$ since $w \notin V$.

Let $v = L(w) \cdot w$. Then for any $x \in H$,

$$L_v(x) = \langle L(w) w, x \rangle = \langle L(w) w, \underbrace{x - \frac{L(x)}{L(w)} w}_{\text{sits inside } \ker(L) = V} + \frac{L(x)}{L(w)} w \rangle$$

$$= L(w) \cdot \frac{L(x)}{L(w)} \langle w, w \rangle = L(x).$$

$\Rightarrow L_v = L$. $\square$

- Given any Banach space V, we denote $V^* = \{\text{all continuous linear functional on } V\}$

For any $L \in V^*$, one can define $\|L\|_{V^*} := \sup_{\|x\|=1} |Tx|$.

Then one can show that $(V^*, \|\cdot\|_{V^*})$ is a Banach space.

Note: If V is a Hilbert space, then

$$\|L_v\|_{V^*} = \sup_{\|w\|=1} |L_v(w)| = \sup_{\|w\|=1} |\langle v, w \rangle| = \|v\|.$$

[we have "$\leq$" by Cauchy-Schwarz; we have "$\geq$" by taking $w = \frac{v}{\|v\|}$]

So what we proved is $H = H^*$, as Hilbert spaces.

In particular, we get

$$(L^2(X))^* \simeq L^2(X). \quad \text{(as Hilbert spaces)}$$