Real Analysis

Last time

- Hilbert space = complete inner-product space ($\Rightarrow$ Banach space $\Rightarrow$ complete metric space)
  Example: $\mathbb{R}^d$, ℓ^2, $L^2(X)$, ...
- Properties of inner-product spaces.
  - Cauchy-Schwarz inequality $|\langle v, w\rangle| \le \|v\| \cdot \|w\|$.
  - Parallelogram law: $\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$
- Properties of Hilbert spaces H
  - Existence of minimizer: $\phi \neq K \subset H$ closed convex $\Rightarrow \exists!\, y \in K$ s.t. $\|x-y\| = \inf\{\|x-z\| : z \in K\}$.
  - Existence of orthogonal complement: $V \subset H$ closed subspace $\Rightarrow H = V \oplus V^\perp$
  - Riesz representation theorem. $L: H \to \mathbb{R}$ continuous linear functional $\Rightarrow \exists!\, v \in H$ s.t. $L = L_v$.
    (i.e. $L(w) = \langle v, w\rangle$, $\forall w$.)

Today: linear functionals on $L^p(X)$

1. Linear functionals on normed vector spaces.

- We have seen.
  Riesz representation thm for $L^2(X)$: $L: L^2(X) \to \mathbb{R}$ continuous linear functional

  $\cancel{\Rightarrow \exists!\, g \in L^2(X) \text{ s.t. } L(f) = \int f g\, d\mu,\ \forall \omega.}$

  $\Rightarrow \exists!\, g \in L^2(X)$ s.t. $L(f) = \int_X f g\, d\mu$, $\forall f \in L^2(X)$.

  Note: If we use complex-valued functions, i.e. $L^2(X, \mathbb{C})$, then $\exists!\, g \in L^2(X, \mathbb{C})$ s.t.
  $$L(f) = \int_X f \bar{g}\, d\mu, \quad \forall f \in L^2(X, \mathbb{C}).$$

  Riesz representation thm for $C_c(X)$: $L: C_c(X) \to \mathbb{R}$ be bounded linear functional

  $\Rightarrow \exists$ Borel measures μ_1, μ_2 on X s.t.
  $$L(f) = \int_X f\, d\mu_1 - \int_X f\, d\mu_2.$$

  Note: We proved this for positive linear functionals in 19 ($\mu_2 = 0$) and then in PSet 10-1 for more general L (with X compact).

  Note: The assumption on X is: X is locally compact metric space, and is σ-compact.
  $\uparrow$ only need: "Hausdorff".

· <u>Justification of assumptions/definitions</u>

(1) The spaces $L^2(X)$, $C_c(X)$ and $L^p(X)$ are normed vector spaces.

$L^2(X)$: $\|f\|_{L^2} = \left(\int_X |f|^2 d\mu\right)^{1/2}$

$C_c(X)$: $\|f\|_{C_c(X)} = \sup_X |f|$

$L^p(X)$: $\|f\|_{L^p(X)} = \left(\int_X |f|^p d\mu\right)^{1/p}$.

(2) Let V be a normed vector space, $\ell: V \to \mathbb{R}$ linear functional. Then

$\underbrace{\ell \text{ is continuous}}_{\substack{\forall \text{ open set } U\subset\mathbb{R},\\ \ell^{-1}(U) \text{ is open.} \\ (a)}} \iff \underbrace{\ell \text{ is continuous at } 0}_{\substack{\forall \varepsilon>0,\ \ell^{-1}((-\varepsilon,\varepsilon)) \text{ contains an open set} \\ (b)}} \iff \underbrace{\ell \text{ is bounded}}_{\substack{\exists C \text{ s.t. } |\ell(v)| \le C\|v\|. \\ (c)}}$

<u>Reason</u>: · Clearly (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (b)

· We also have (b) $\Rightarrow$ (a): Let $v \in \ell^{-1}(U)$. Then $\ell(v) \in U \Rightarrow \exists \varepsilon$ s.t. $(\ell(v)-\varepsilon, \ell(v)+\varepsilon) \subset U$.

$\Rightarrow$ By triangle inequality, $\underbrace{v + \ell^{-1}((-\varepsilon,\varepsilon))}_{\text{open}} \subset \ell^{-1}(U)$.

· Finally (a) $\Rightarrow$ (c): By (a), $\ell^{-1}((-1,1))$ is open.

Since $0 \in \ell^{-1}((-1,1))$, so $\exists r>0$ s.t. $B_r(0) \subset \ell^{-1}((-1,1))$

i.e. $\ell(B_r(0)) \subset (-1,1)$ $\Rightarrow$ $|\ell(v)| = \frac{\|v\|}{r} \ell\left(\frac{v \cdot r}{\|v\|}\right) \le \frac{\|v\|}{r}$. $\square$

(3). Any $\underbrace{\text{positive linear functional}}_{f\ge 0 \Rightarrow \ell(f) \ge 0}$ $\ell: C(X) \to \mathbb{R}$ is bounded if X is compact

<u>Reason</u>: Let $C = |\ell(1)|$. Then for any $f \in C(X)$, $|f| \le 1$, we have

$1 \pm f \ge 0 \Rightarrow C \pm \ell(f) \ge 0 \Rightarrow |\ell(f)| \le C$.

So for any $f \in C(X)$ we have

$|\ell(f)| = \left|\ell\left(\frac{f}{\|f\|}\right) \cdot \|f\|\right| = \|f\| \cdot \ell\left(\frac{f}{\|f\|}\right) \le \|f\| \cdot C$. $\square$

<u>Note</u>: By the same argument one can prove that if X is locally compact metric (or Hausdorff), then for any compact set $K \subset X$, $\exists C_K$ s.t.

$|\ell(f)| \le C_K \|f\|$, $\forall f \in C_c(X)$ with $\operatorname{supp}(f) \subset K$

for positive linear functional ℓ.

2. Linear functionals on $L^p(X)$

- Now consider the space $L^p(X)$, $p \geq 1$.

  As in the case of $L^2(X)$, one can start with a function g, and define

$$L_g : L^p(X) \to \mathbb{R}, \qquad f \longmapsto \int_X f g \, d\mu.$$

  Note that for L_g to be well-defined, we need $fg \in L^1(X)$ for all $f \in L^p(X)$.

  This is the case if $g \in L^q(X)$, where $\frac{1}{q} + \frac{1}{p} = 1$.

  In fact, we have

<u>Prop</u>: For any $g \in L^q(X)$, $L_g : L^p(X) \to \mathbb{R}$ is a continuous linear functional.

<u>Proof</u>: We have seen that L_g is well-defined.

It is obviously linear, i.e. $L_g(c_1 f_1 + c_2 f_2) = c_1 L_g(f_1) + c_2 L_g(f_2)$.

To prove L_g is continuous, it is enough to prove L_g is bounded, which follows from the Hölder's inequality

$$|L_g(f)| \leq \|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q} = C \|f\|_{L^p}, \quad C = \|g\|_{L^q}. \qquad \square$$

- The main theorem we want to prove today is

<u>Thm</u>: (Riesz representation thm for $L^p(X)$).

Suppose μ is a σ-finite measure on X. (i.e. $X = \bigcup_{n=1}^{\infty} X_n$, and $\mu(X_n) < \infty$, $\forall n$)

Let $1 \leq p < \infty$, and $q \geq 1$ be s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

Then for any continuous linear functional $l : L^p(X) \to \mathbb{R}$, there exists a unique $g \in L^q(X)$ s.t. $l = L_g$.

<u>Rmk</u>:
- The thm fails for $p = +\infty$.

  In other words, for any $g \in L^1(X)$, g defines a continuous linear functional $L_g : L^\infty(X) \to \mathbb{R}$. However, there are more linear functionals on $L^\infty(X)$ continuous that these L_g's.

- The main ingredient in the proof is the following theorem that we will prove later.

<u>Radon-Nikodym Thm</u>: Let μ be a σ-finite measure on $(X, \mathcal{F})$.

Let ν be a signed measure on $(X, \mathcal{F})$ s.t. $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Then $\exists f \in L^1(X)$ s.t. $\nu(A) = \int_A f \, d\mu$.

i.e. $\nu : \mathcal{F} \to \overline{\mathbb{R}}$ s.t.
- $\nu(\phi) = 0$,
- $\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$, disjoint

· Proof of Riesz representation thm for $L^p(X)$

Step 1. $\boxed{\text{Uniqueness}}$ Let $g_1, g_2 \in L^q(X)$ be s.t. $L_{g_1} = L_{g_2} \Rightarrow L_{g_1 - g_2} = 0$ on $L^p(X)$.

Let $g = g_1 - g_2 \in L^q(X)$. Suppose $g \neq 0$ in $L^q(X)$, then $\exists$ measurable set $A \subset X$ with $0 < \mu(A) < \infty$, s.t. $g > 0$ on A or $g < 0$ on A.

Let $f = \chi_A \in L^p(X)$. Then $L_g(f) = \int_X fg\, d\mu \neq 0$, a contradiction.

$\boxed{\text{Step 2}}$ $\boxed{\text{Suppose } \mu(X) < \infty}$

[Idea: linear functional $\rightsquigarrow$ signed measure $\xrightarrow{R\text{-}N}$ a function $g \in L^1 \longrightarrow g \in L^q$]

· Since $\mu(X) < \infty$, for any measurable set $A \in \mathcal{F}$, the function $\chi_A \in L^p(X)$.

So we can define $\nu : \mathcal{F} \to \mathbb{R}$ by

$$\nu(A) := l(\chi_A).$$

This is a signed measure since

① $\nu(\emptyset) = l(0) = 0$.

② Let $A_1, A_2, \cdots \in \mathcal{F}$ be disjoint. Then $\chi_{\bigcup_{i=1}^k A_i} \to \chi_{\bigcup_{i=1}^\infty A_n}$ pointwise.

But we also have $\left| \chi_{\bigcup_{i=1}^k A_i} \right| \leq 1 \in L^p(X)$.

So by dominated convergence theorem, $\chi_{\bigcup_{i=1}^k A_i} \to \chi_{\bigcup_{i=1}^\infty A_i}$ in $L^p(X)$.

Since l is continuous, we get

$$\sum_{i=1}^k l(\chi_{A_i}) = l(\chi_{\bigcup_{i=1}^k A_i}) \longrightarrow l(\chi_{\bigcup_{i=1}^\infty A_i})$$

i.e. $\nu(\bigcup_{i=1}^\infty A_i) = \lim_{k \to \infty} \sum_{i=1}^k \nu(A_i)$.

Note: $\mu(A) = 0 \Rightarrow \chi_A = 0$ in $L^p(X)$
$\Rightarrow \nu(A) = l(\chi_A) = 0$

· By Radon-Nikodym theorem, $\exists g \in L^1(X)$ s.t. $\nu(A) = \int_A g\, d\mu, \ \forall A \in \mathcal{F}$.

· We want to prove $l = L_g$ on $L^p(X)$:

By def., for any $A \in \mathcal{F}$, one has $l(\chi_A) = \nu(A) = \int_A g\, d\mu = L_g(\chi_A)$.

By linearity, $l = L_g$ for all simple function.

But in lec. 21, we have shown that the space of simple functions is dense in $L^p(X)$.

~~Fact: $L_g : L^p(X) \to \mathbb{R}$ is well-defined and is continuous.~~

Reason: Assume $g \geq 0$. otherwise consider g_+ and g_-.

Let $f \in L^p(X)$. Assume $f \geq 0$. Otherwise consider f_+ and f_-.

Then $\exists$ simple functions $\varphi_1, \varphi_2, \cdots \to f$ in $L^p(X)$. Assume $|f| \in C$. Then f

By ~~monotone~~ dominated convergence, $L_g(f) = \lim_{n \to \infty} L_g(\varphi_n) = \lim_{n \to \infty} l(\varphi_n) = l(f) < \infty$. $\Rightarrow L_g(f)$ well-defined ✓.

For unbounded $f \in L^p(X)$, $f \geq 0$, we let $f_N = \min\{f, N\}$.

Then by monotone convergence, $L_g(f) = \lim_{N \to \infty} L_g(f_N) = \lim_{N \to \infty} l(f_N) = l(f) < \infty$.

$\boxed{l = L_g \text{ on } L^p(X).}$

• It remains to prove $g \in L^q$. WLOG, we assume $g \geq 0$, otherwise we work on g_+ and g_-.

Since $l = L_g$ is continuous, it is a bounded linear functional. So $\exists C > 0$ s.t.

$$|L_g(f)| \leq C \|f\|_{L^p(X)}, \quad \forall f \in L^p(X).$$

Case 1: $1 < p < \infty$.

Idea: Want $\|g\|_{L^q} < \infty$. So we take $f = g^{q-1}$, so that formally

$$\|g\|_{L^q}^q = |L_g(f)| \leq C \cdot \|f\|_{L^p}.$$

and $\|f\|_{L^p} = \left(\int_X |g|^{pq-p} \, d\mu\right)^{1/p} = \left(\int_X |g|^q \, d\mu\right)^{1-\frac{1}{q}} = \|g\|_{L^q}^{q-1}$

$\Rightarrow \|g\|_{L^q} \leq C.$

But this argument is NOT true since we don't know $f = g^{q-1} \in L^p$ yet.

Let $f_N = \min(g, N)^{q-1}$. Then $f_N \in L^p(X)$ since $\mu(X) < +\infty$.

We have. $L_g(f_N) = \int_X g \cdot f_N \, d\mu \geq \int_X \min(g,N)^q \, d\mu = \|\min(g,N)\|_{L^q}^q$

$\cdot \|f_N\|_{L^p} = \left(\int_X \min(g,N)^{q} \, d\mu\right)^{1/p} = \|\min(g,N)\|_{L^q}^{q-1}$

Since $|L_g(f_N)| \leq C \|f_N\|_{L^p}$ we get $\|\min(g,N)\|_{L^q} \leq C$.

Letting $N \to \infty$ and using monotone convergence, we get

$$\|g\|_{L^q} \leq C.$$

Case 2, $p = 1$.

Let $f_N = \chi_{g>N}$. Then $f_N \in L^p(X)$. So

$$N \int_{g>N} 1 \, d\mu \leq \int_{g>N} g \, d\mu = \int_X g f_N \, d\mu \leq C \|f_N\|_{L^1} = C \int_{g>N} 1 \, d\mu.$$

$\Rightarrow$ For $N > C$, $\mu(\{g > N\}) = 0$. So $\|g\|_{L^\infty} \leq C$.

$\boxed{\text{Step 3}}$ $\boxed{\text{Suppose } \mu(X) = +\infty}$

Some details are missing. Can you write down a complete proof of step 3?

By assumption, $\exists X_1 \subset X_2 \subset \cdots \subset X$, $\mu(X_n) < \infty$, s.t. $X = \bigcup_{n=1}^{\infty} X_n$.

By step 2, one can find $g_n \in L^q(X_n)$ s.t. $l = L_{g_n}$ on $L^p(X_n)$.

By step 1, for $m < n$ one has $g_m = g_n$ on X_m. Thus one get one function g defined on X s.t. $g = g_n$ on X_n.

In Step 2, we showed that $\|g_n\|_{L^q(X_n)} \leq C$ for the same C, $\forall n$.

So by monotone convergence argument, $g \in L^q(X)$ and $\|g\|_{L^q} \leq C$. □