Real Analysis

Last time.

- Continuous linear functional: continuous = continuous at 0 = bounded $\leftrightarrow$ positive.
- Riesz representation theorem (L^p). $\boxed{(L^p)^* = L^q} \longleftrightarrow$ Suppose μ is a σ-finite measure on X. Suppose $1 \le p < \infty$, $1 < q \le +\infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then any linear functional continuous $l: L^p(X) \to \mathbb{R}$ is of the form $l(f) = L_g(f) = \int_X f g \, d\mu$ for some $g \in L^q(X)$.

Main ingredient: Radon-Nikodym Theorem.

Today. Radon-Nikodym-Lebesgue Theorem

1. Signed measure

- Recall: In Lecture 10, 11, 16 we have seen that the "countable additivity w.r.t. domain"

$$\int_{\bigcup_{n=1}^{\infty} A_n} f(x)\, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f(x)\, d\mu \qquad (A_n \in \mathcal{F}, \text{ disjoint})$$

$\Rightarrow$ • If f is non-negative, then $\mu_f =$ "$f d\mu$" defines a measure on $\mathcal{F}$ via

$$m(A) = \int_A f \, d\mu, \qquad \forall A \in \mathcal{F}.$$

• If f is absolutely integrable, then $\mu_f = f\, d\mu$ defines a "signed measure".
[In the sense that one still has "countable additivity", but one losts the "non-negativity", i.e. one may have $m(A) < 0$.]

Note: • The measure of a set could be $+\infty$.
So the signed measure of a set could be $\pm\infty$.
In other words, we don't really need f to be absolutely integrable.
• On the other hand side, we don't want a signed measure to take both the value $+\infty$ and the value $-\infty$:
if $\mu(A) = +\infty$, $\mu(B) = -\infty$, what can $\mu(A \cup B)$ be?

• This motivates

Def.: A signed measure on a measurable space $(X, \mathcal{F})$ is a map $\mu: \mathcal{F} \to [-\infty, +\infty]$ s.t.
(1) $\mu(\phi) = 0$.
(2) $\mu(\mathcal{F}) \subset [-\infty, +\infty)$ or $\mu(\mathcal{F}) \subset (-\infty, +\infty]$.
(3) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ if $A_n \in \mathcal{F}$ are disjoint, where we assume the series (in the RHS) converges absolutely if the measure in the LHS is finite.

Examples: (1) If μ_1, μ_2 are measures, and either μ_1 or μ_2 is a finite measure, then $\mu = \mu_1 - \mu_2$ is a signed measure.
(2) If f is measurable w.r.t. a measure μ, and either f_+ or f_- is integrable, then $\mu_f =$ "$f d\mu$" is a signed measure.
(c.f. Lec. 11, page 1)

· ***Def.*** Let μ be a signed measure on $(X, \mathcal{F})$.

(1) We say a set $A \in \mathcal{F}$ is a <u>positive set</u> if $\forall$ measurable set $B \subset A$, we have $\mu(B) \geq 0$.

(2) - - — — — - <u>negative set</u> . . - — — — — — - - - - $\mu(B) \leq 0$.

(3) If $A \in \mathcal{F}$ is both positive and negative, we say A is a <u>null set</u>.

<u>Note</u>: A is null means $\mu(B) = 0$ for any measurable $B \subset A$.
This is different from simply requiring $\mu(A) = 0$.

For any function $f: X \to [-\infty, +\infty]$, one can decompose X into $X_+ \cup X_-$ s.t. f is $\begin{cases} \geq 0 & \text{on } X_+ \\ < 0 & \text{on } X_- \end{cases}$

Similarly, for signed measures, we have

<u>Thm</u>. (The Hahn decomposition theorem) Let μ be a signed measure on $(X, \mathcal{F})$.

Then $\exists$ positive set X_+ and negative set X_- for μ s.t. $X = X_+ \cup X_-$ and $X_+ \cap X_- = \phi$. Moreover, if X_+', X_-' is another such decomposition of X, then $X_+ \triangle X_+'$ is null. [<u>Note</u>: $X_+ \triangle X_+' = X_- \triangle X_-'$]

<u>Proof</u>: WLOG, we may assume $\mu(A) < +\infty$, $\forall A \in \mathcal{F}$.

[Idea: Pick X_+ to be the positive set of maximal measure]

· Let $m_+ = \sup\{\mu(A) : A \text{ is a positive set}\}$. $< +\infty$. $\leftarrow \boxed{\text{why?!}}$

Let $A_1, A_2, \cdots$ be a sequence of positive sets s.t. $\mu(A_n) \to m_+$.

Let $A = \bigcup_{n=1}^{\infty} A_n$. Then A is positive. $\leftarrow \boxed{\text{Why?}}$ and by (todays) PSet 13-1, problem 1,

$$m_+ \geq \mu(A) = \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} A_n\right) \geq \lim_{N\to\infty} \mu(A_N) = m_+.$$

So we get $m_+ = \mu(A) < +\infty$.

· Let $X_+ = A$, $X_- = X \setminus X_+$. We need to show that X_- is a negative set.

– Suppose X_- is NOT a negative set, then $\exists B_1 \subset X_-$ s.t. $\mu(B_1) > 0$.

If B_1 is a positive set, then $X_+ \cup B_1$ is a positive set with $\mu(X_+ \cup B_1) > m_+$, contradiction.

So B_1 is NOT a positive set. In other words, $\exists B_2 \subset B_1$ s.t. $\mu(B_2) > \mu(B_1)$.

$\Rightarrow \exists n_1$ s.t. $\mu(B_2) \geq \mu(B_1) + \frac{1}{n_1}$.

<u>Greedy choice</u>: We will pick B_2, n_1 s.t. n_1 is the smallest positive integer s.t. B_2 exists.

We continue this process to get $B_3, n_2, B_4, n_3, \cdots$ s.t.

$$B_1 \supset B_2 \supset B_3 \supset \cdots$$

and

$$\mu(B_{k+1}) \geq \mu(B_k) + \frac{1}{n_k} \geq \mu(B_{k-1}) + \frac{1}{n_{k-1}} + \frac{1}{n_k} \geq \cdots \geq \sum_{i=1}^{k} \frac{1}{n_i}.$$

Since $\mu(B_1) < +\infty$, by PSet 13-1 Problem 1, if we write $B = \bigcap_{k=1}^{\infty} B_k$, then

$$+\infty > \mu(B) = \lim_{k\to\infty} \mu(B_k) \geq \sum_{i=1}^{\infty} \frac{1}{n_i} > 0.$$

This implies $\lim_{i\to\infty} n_i = +\infty$.

~~In particular, one can find k large s.t. $n_k \geq 2 n_1$~~

– We claim that B is a positive set in $X_-^\bullet$ with $\mu(B)>0$. This contradicts with the choice of $X_+^\bullet$, so we are done.

Again we argue by contradiction. Suppose B is NOT a positive set, then $\exists C \subset B$ s.t.

$$\mu(C) > \mu(B) = \lim_{k\to\infty} \mu(B_k)$$

So $\exists m_0>0$ and $N_0>0$ s.t. $\mu(C)-\mu(B_k) > \frac{1}{m_0}$ for $\forall k > N_0$.

However, since $\lim_{i\to\infty} n_i = +\infty$, one can find k s.t. $n_k > m_0$.

This contradicts with the choice of B_{k+1}, n_k: n_k is the smallest ~~number~~ positive integer s.t. one can find $B_{k+1}\subset B_k$ and $\mu(B_{k+1}) \geq \mu(B_k) + \frac{1}{n_k}$ $\square$

• As a consequence, we get

<u>Thm.</u> (Jordan decomposition theorem) Let μ be a signed measure on $(X,\mathcal{F})$. Then $\exists$ unique (positive) measures μ_+, μ_- s.t. $\mu=\mu_+-\mu_-$. Moreover, $\mu_+|_{X_-}=0$, $\mu_-|_{X_+}=0$ for a Hahn decomposition $X=X_+\cup X_-$.

The expression $\mu_+|_{X_-}=0$ means: $\forall A\subset X_-$, $\mu_+(A)=0$

<u>Proof</u>: • Let $X=X_+^\bullet \cup X_-$ be the Hahn decomposition. Define for $\forall A\in\mathcal{F}$,

$$\mu_+(A)=\mu(A\cap X_+), \qquad \mu_-(A)=-\mu(A\cap X_-).$$

Since X_+ is a positive set, X_- is a negative set, both μ_+ and μ_- are (positive) measures.

By definition, we have $\mu=\mu_+-\mu_-$ and $\mu_+|_{X_-^\bullet}=0$, $\mu_-|_{X_+}=0$.

• To prove the uniqueness, suppose $\mu=\nu_+-\nu_-$, where ν_+ and ν_- are (positive) measures, and $\nu_+|_{X_-'}=0$, $\nu_-|_{X_+'}=0$, $X=X_+'\cup X_-'$ is a Hahn decomposition.

Then $\forall A\subset X_+'$, we have $\mu(A)=\nu_+(A)$

$\forall B\subset X_-'$, — — $\mu(B)=\nu_-(B)$

Since $X_+\triangle X_+'$ is a null set, and since $\forall A\in\mathcal{F} \Rightarrow (A\cap X_+)\triangle(A\cap X_+')\subset X_+\triangle X_+'$, we get for $\forall A\in\mathcal{F}$,

$$\mu_+^\bullet(A)=\mu(A\cap X_+)=\mu(A\cap X_+')=\nu_+(A\cap X_+')=\nu_+(A).$$

$$\mu_-(A)=-\mu(A\cap X_-)=-\mu(A\cap X_-')=\nu_-(A\cap X_-')=\nu_-(A). \quad \square$$

• Finally suppose $X=X_+'\cup X_-'$ is another such decomposition.

Then $X_+\setminus X_+' = X_+\cap(X_+')^c = X_+\cap X_-'$

$\Rightarrow$ $X_+\setminus X_+'$ is both positive and negative, i.e. $X_+\setminus X_+'$ is null.

Similarly $X_+'\setminus X_+$ is null. So $X_+\triangle X_+'$ is null.

· <u>Remark</u>: · In the Jordan decomposition $\mu = \mu_+ - \mu_-$, we will call μ_+ and μ_- the <u>positive/negative part</u>, or the <u>positive/negative variation</u> of μ.

· Inspired by the decomposition $f = f_+ - f_-$ and $|f| = f_+ + f_-$ of functions, we will call

$$|\mu| = \mu_+ + \mu_-$$

the <u>absolute value</u> or the <u>total variation</u> of μ.

· We call a signed measure μ is a <u>finite</u>/<u>σ-finite</u>, if $|\mu|$ is finite / σ-finite.

<u>Note</u>: μ is finite means $|\mu(A)| \le C$ for some constant C, $\forall A \in \mathcal{F}$.

2. The Lebesgue-Radon-Nikodym Theorem

· Let μ, ν be two signed measures on $(X, \mathcal{F})$.

<u>Def.</u> We say μ and ν are <u>mutually singular</u>, and write $\mu \perp \nu$, if $\exists$ decomposition $X = X_1 \cup X_2$, $X_1 \cap X_2 = \phi$, s.t. X_2 is null for μ_1, and X_1 is null for μ_2.

X_2 is null for μ_1 $\Leftrightarrow$ μ_1 is supported on X_1

X_1 is null for μ_2 $\Leftrightarrow$ μ_2 is supported on X_2

<u>Example</u>: · In Jordan decomposition $\mu = \mu_+ - \mu_-$, we have $\mu_+ \perp \mu_-$.

· For any delta measure μ_{x_0} on $\mathbb{R}^d$, we have $m \perp \mu_{x_0}$

m: $\boxed{\text{the Lebesgue measure}}$

· Let μ, ν be signed measures on $(X, \mathcal{F})$.

<u>Def.</u> We say ν is <u>absolutely continuous</u> w.r.t. μ, and write $\nu \ll \mu$, if every μ-null set is also a ν-null set.

<u>Example</u>: Suppose f is measurable and a.e. finite. Let $\mu_f = f\,d\mu$.

Then $\mu_f \ll \mu$.

<u>Reason</u>: If $\mu(A) = 0$, then $f \cdot \chi_A = 0$ a.e.

$\Rightarrow \mu_f(A) = \int_A f\,d\mu = \int_X f\chi_A\,d\mu = 0$.

<u>Prop.</u> Let ν be a finite signed measure, and let μ be a (positive) measure on $(X, \mathcal{F})$. Then $\nu \ll \mu$ if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall A \in \mathcal{F}$, if $\mu(A) < \delta$, then $|\nu(A)| < \varepsilon$.

[Compare: "Absolute continuity" in Lecture 11, page 5:

$f \in L^1(A) \Rightarrow \forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall B \subset A$, $m(B) < \delta \Rightarrow \mu_{|f|}(B) = \int_B |f|\,dx < \varepsilon$.]

Proof will be left as an exercise

· Given any measure μ and signed measure ν, we want to know: when can we find a measurable function f s.t. $\nu = \mu_f$.

Note that this "should" require $\nu \ll \mu$.

Moreover, if $\nu \perp \mu$, then one can never has $\nu = \mu_f$ unless $\nu \equiv 0$.

The main theorem is

<u>Thm.</u> (The Lebesgue-Radon-Nikodym Theorem)

> Let μ be a σ-finite measure, and ν a σ-finite signed measure, on $(X, \mathcal{M})$.
> Then there exists unique signed measures μ_f and μ_s s.t.
> $$\nu = \mu_f + \mu_s,$$
> where $\mu_f \perp \mu_s$ and $\mu_f \ll \mu$. Moreover, ① $\exists$ extended integrable function f on X s.t. $\mu_r = \mu_f = f\,d\mu$. ② μ_r is a measure iff $f \geq 0$
> ③ μ_r is finite iff $f \in L^1(X, \mu)$.

f is extended integrable:
- f is measurable
- either $f_+ \in L^1$, or $f_- \in L^1$

As a corollary, we get

<u>Cor.</u> (The Radon-Nikodym Theorem)

> Let μ be a σ-finite measure, and ν a σ-finite signed measure on $(X, \mathcal{M})$
> Suppose $\nu \ll \mu$. Then $\exists$ extended integrable function f s.t. $\nu = \mu_f$.
> Moreover, if ν is finite, then $f \in L^1(X, \mu)$.

Before we prove the theorem, we first prove a technical lemma.

<u>Lemma.</u>

> Let μ, ν be finite (positive) measures on $(X, \mathcal{M})$. Then either $\nu \perp \mu$, or $\exists \varepsilon > 0$ and $A \in \mathcal{M}$ s.t. $\nu(A) > 0$, and A is a positive set for the signed measure $\mu - \varepsilon\nu$.

<u>Proof.</u> Consider the sequence of signed measures $\mu - \frac{1}{n}\nu$.

Denote the corresponding Hahn decompositions by $X = X_n^+ \cup X_n^-$.

Let $X^+ = \bigcup_n X_n^+$, $X^- = \bigcap_n X_n^- = (X^+)^c$.

Then $\forall n$, $X^- \subset X_n^- \Rightarrow X^-$ is a negative set for $\mu - \frac{1}{n}\nu$, i.e.

$$0 \leq \mu(X^-) \leq \frac{1}{n}\nu(X^-), \quad \forall n$$

Letting $n \to \infty$, we get $\mu(X^-) = 0$.

<u>Case 1</u>: $\nu(X^+) = 0$. Then $\nu \perp \mu$.

<u>Case 2</u>: $\nu(X^+) > 0$. Since $\nu(X^+) = \lim_{N \to \infty} \nu\left(\bigcup_{n=1}^{N} X_n^+\right)$, one can find X_n^+ s.t. $\nu(X_n^+) > 0$. Then for $\varepsilon = \frac{1}{n}$, the set $A := X_n^+$ is a positive set for $\mu - \frac{1}{n}\nu$ by def. $\square$

Proof of the Lebesgue-Radon-Nikodym Theorem

Case 1: ν is finite and positive.

- Define $\mathcal{M} = \{ f : X \to [0,+\infty] : \int_A f \, d\mu \le \nu(A), \ \forall A \in \mathcal{F} \}$

Then
- $\mathcal{M} \neq \emptyset$ since $0 \in \mathcal{M}$.
- Suppose $f, g \in \mathcal{M}$, then $\max(f,g) \in \mathcal{M}$, since

  (for $B = \{x : f > g\}$ and for $\forall A \in \mathcal{F}$,

$$\int_A \max(f,g)\, d\mu = \int_{A\cap B} f \, d\mu + \int_{A \cap B^c} g \, d\mu \le \nu(A\cap B) + \nu(A \cap B^c) = \nu(A).)$$

- Now set

$$m := \sup_{f \in \mathcal{M}} \int_X f \, d\mu \le \nu(X) < +\infty.$$

Let $f_n \in \mathcal{M}$ be s.t. $\int_X f_n \, d\mu \to m$. Let $g_n = \sup\{f_1, \dots, f_n\} \in \mathcal{M}$.

Then $g_n \nearrow f := \sup_n f_n$. So

$$m = \lim_{n\to\infty} \int_X f_n \, d\mu \le \lim_{n\to\infty} \int_X g_n \, d\mu \le m.$$

It follows from the monotone convergence thm that

$$m = \lim_{n\to\infty} \int_X g_n \, d\mu = \int_X f \, d\mu. \quad (\Rightarrow f \text{ is a.e. finite})$$

For any $A \in \mathcal{F}$, we have by monotone convergence, $\int_A f \, d\mu = \lim_{n\to\infty} \int_A g_n \, d\mu \le \nu(A)$. So $f \in \mathcal{M}$.

WLOG, we assume $f < \infty$. Then we get $\int_X f \, d\mu \le \nu(X)$.

- Now consider the measure $\mu_s = \nu - f\, d\mu$. [It is a measure since $f \in \mathcal{M}$]

Suppose μ_s is NOT singular w.r.t. μ. Then by lemma, $\exists\, \varepsilon > 0$ and $A \in \mathcal{F}$ s.t. $\mu(A) > 0$, and $\mu_s(B) > \varepsilon \mu(B)$ for $\forall B \subset A$

In particular, for $\forall C \in \mathcal{F}$,

$$\begin{aligned}
\int_C (f + \varepsilon \chi_A)\, d\mu &= \int_C f \, d\mu + \varepsilon \cdot \mu(A \cap C) \\
&\le \int_C f \, d\mu + \nu(A\cap C) - \int_{A \cap C} f \, d\mu \\
&= \int_{C \cap A^c} f \, d\mu + \nu(A \cap C) \\
&\le \nu(C \cap A^c) + \nu(C \cap A) \\
&= \nu(C)
\end{aligned}$$

So $f + \varepsilon \chi_A \in \mathcal{M}$, i.e.

$$m \ge \int (f + \varepsilon\chi_A)\, d\mu = \int f \, d\mu + \varepsilon \mu(A) > m$$

a contradiction. Thus we get

$$\mu_s \perp \mu.$$

- Let $\mu_r = \nu - \mu_s = f\,d\mu$. Then we get the desired decomposition
$$\nu = \mu_r + \mu_s, \quad \mu_s \perp \mu, \quad \mu_r \ll \mu.$$
Moreover, $f \geq 0$, and $f \in L^1(X,\mu)$ since $\int_X f\,d\mu \leq \nu(X) < +\infty$.

- To prove the uniqueness, we suppose
$$\nu = \mu_r' + \mu_s', \quad \mu_s' \perp \mu, \quad \mu_r' \ll \mu.$$
Then $\mu_r' - \mu_r = \mu_s - \mu_s'$. Since $\mu_r' - \mu_r \ll \mu$, $\mu_s - \mu_s' \perp \mu$, we must have $\mu_r' - \mu_r = 0 = \mu_s - \mu_s'$, i.e. $\mu_r = \mu_r'$, $\mu_s = \mu_s'$.

Case 2: μ, ν are σ-finite (positive) measures.

- We write $X = \bigcup_{n=1}^{\infty} X_n$, where $\mu(X_n) < +\infty$, $\nu(X_n) < +\infty$, and X_n's are disjoint $\boxed{\text{How?}}$

Define two sequences of measures μ_n, ν_n on $\mathcal{F}$ via
$$\mu_n(A) := \mu(A \cap X_n), \qquad \nu_n(A) := \nu(A \cap X_n).$$
They are finite measures. So one has
$$\nu_n = \mu_n^r + \mu_n^s, \quad \mu_n^r \ll \mu_n, \quad \mu_n^s \perp \mu_n, \quad \text{and } \mu_n^r = f_n\,d\mu_n, \quad f_n \geq 0, \quad f_n \in L^1(X,\mu)$$
By definition, $\mu_n(X_n^c) = \nu_n(X_n^c) = 0$. So we may assume $f_n = 0$ on X_n^c [so $f_n\,d\mu_n = f_n\,d\mu$]

Let $f = \sum_{n=1}^{\infty} f_n$, $\mu_r = \sum \mu_n^r = f\,d\mu$, and let $\mu_s = \sum_n \mu_n^s$.

One can check
- $\mu_s \perp \mu$, $\mu_r \ll \mu$.
- $\nu = \mu_r + \mu_s$.

Case 3: ν is only σ-finite signed measure.

- Write $\nu = \nu^+ - \nu^-$, where one of ν^+, ν^- is a finite measure

We have $\nu^{\pm} = \mu_r^{\pm} + \mu_s^{\pm}$. Let $\mu_r = \mu_r^+ - \mu_r^-$, $\mu_s = \mu_s^+ - \mu_s^-$.

Then $\mu_r^{\pm} = f^{\pm}\,d\mu$, and either f^+ or $f^- \in L^1$.

$\Rightarrow f = f^+ - f^-$ is extended integrable.

- Uniqueness:

Write $X = \bigcup_{n=1}^{\infty} X_n$, $|\nu|(X_n) < +\infty$, and X_n's are disjoint.

Again, we can define $\nu_n(A) = \nu(A \cap X_n)$ to get finite signed measure ν_n's.

Note: If $\nu = \mu_r + \mu_s$, then $\nu_n = \mu_r^n + \mu_s^n$, where $\mu_r^n(A) = \mu_r(A \cap X_n)$, $\mu_s^n(A) = \mu_s(A \cap X_n)$.

By uniqueness in Case 1, these μ_r^n, μ_s^n are unique.

Now let $\nu = \tilde{\mu}_r + \tilde{\mu}_s$. then $\nu_n = \tilde{\mu}_r^n + \tilde{\mu}_s^n$. So we must have
$$\tilde{\mu}_r^n = \mu_r^n, \quad \tilde{\mu}_s^n = \mu_s^n$$
$$\Rightarrow \tilde{\mu}_r = \mu_r, \quad \tilde{\mu}_s = \mu_s. \qquad \square$$

Notation: If $d\nu = f\,d\mu$, we say f is the Radon-Nikodym derivative of ν w.r.t. μ since formally we have
$$\frac{d\nu}{d\mu} = f.$$