Real Analysis

Lecture 25
05/31/2018

Last time.

- Signed measure $\mu: \mathcal{F} \to [-\infty, +\infty]$ (can't take both $+\infty$ and $-\infty$)
  - Example: $\mu_1 - \mu_2$ for positive measures μ_1, μ_2, one of which is finite.
    [Compare: "positive numbers" to "signed numbers".]
- Jordan decomposition: μ is signed measure $\Rightarrow \mu = \mu_+ - \mu_-$, μ_+ & μ_- are measures
  [compare: $f = f_+ - f_-$]

  Hahn decomposition: μ is a signed measure $\Rightarrow X = X_+ \cup X_-$, $X_+ \cap X_- = \phi$

  positive set　　negative set.
- Mutually singular: $\mu \perp \nu \Leftrightarrow X = X_1 \cup X_2$ s.t. $\mu|_{X_2} = 0$, $\nu|_{X_1} = 0$.

  Absolute continuous: $\nu \ll \mu \Leftrightarrow$ Any μ-null set is also a ν-null set.
- The Lebesgue-Radon-Nikodym theorem:

  > Let μ be a σ-finite measure, and ν a σ-finite signed measure.
  > Then there exist unique signed measures μ_r and μ_s s.t.
  > $$\nu = \mu_r + \mu_s, \quad \mu_r \ll \mu, \quad \mu_s \perp \mu.$$
  > Moreover,
  > ① $\exists$ integrable function (in the extended sense) f s.t. $\mu_r = f\,d\mu$.
  > ② ν is a measure $\Leftrightarrow$ μ_r, μ_s are measures. ($\Rightarrow f \geq 0$)
  > ③ ν is finite $\Rightarrow$ μ_r is finite $\Leftrightarrow f \in L^1(X, \mu)$

Today. The Lebesgue differentiation theorem

1. The dimension 1 case

- A special case of the Lebesgue-Radon-Nikodym theorem is
  $$\nu \ll \mu \Rightarrow \exists f \text{ s.t. } d\nu = f\,d\mu. \quad (f \in L^1 \text{ or in the extended sense})$$

The function f is called the Radon-Nikodym derivative of ν w.r.t. μ.

We would like to explain the reason why we call f "derivative".

It is not just because formally we have "$f = \dfrac{d\nu}{d\mu}$".

We always assume the measure $= m$.

- We start with a very simple case.

Recall: If $F: [a,b] \to \mathbb{R}$ is a function. We say a function f is ~~the~~ derivative of F if for $x \in (a,b)$, one has

$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

In calculus we have learned that if f is continuous, then f is the derivative of the function

$$F(x) = \int_a^x f(t)\,dt.$$

In other words ~~Alternatively~~, one has

$$\lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t)\,dt = f(x) = \lim_{h \to 0^-} \int_{x-h}^{x} f(t)\,dt.$$

But: what if f is only a function in L^1?

Thm. (Lebesgue differentiation theorem, dim $= 1$).

Suppose $f \in L^1(\mathbb{R})$. Then

$$\lim_{h \to 0^+} \frac{1}{h} \int_{[x,x+h]} f(t)\,dt = f(x), \quad \text{a.e. } x \in \mathbb{R}$$

$$\lim_{h \to 0^+} \frac{1}{h} \int_{[x-h,x]} f(t)\,dt = f(x), \quad \text{a.e. } x \in \mathbb{R}$$

[In other words, if we let $F(x) = \int_{[-\infty, x]} f(t)\,dt$.
Then $F'(x) = f(x)$, a.e. $x \in \mathbb{R}$.]

The idea of proof (of many theorems): "density argument"

To prove a theorem holds for a class of functions

- First prove the theorem holds for a subclass of "nice" functions which is dense in the desired class
- Then prove the quantity (e.g. $\frac{1}{h}\int_{[x,x+h]} |f-g|(t)\,dt$) is controlled by some "norm" of $f-g$.
- Try to combine the two steps to a proof for all functions.

For this theorem, of course the "dense subclass" can be chosen to be all continuous functions. The "Step 2" that we need is

Thm. (One-side Hardy-Littlewood maximal inequality) Suppose $f \in L^1(\mathbb{R}, m)$.

Then for any $\alpha > 0$,

$$m\left(\left\{x \in \mathbb{R} : \sup_{h>0} \frac{1}{h} \int_{[x,x+h]} |f(t)|\,dt > \alpha\right\}\right) \le \frac{1}{\alpha} \int_{\mathbb{R}} |f(t)|\,dt.$$

We will NOT prove this. But instead, we will prove a higher dimensional analogue

Proof of the Lebesgue differentiation theorem, dim = 1 case:

- We want to prove that for any $f \in L^1(\mathbb{R})$,

$$\lim_{h \to 0+} \frac{1}{h} \int_{[x, x+h]} f(t)\, dt = f(x) \quad , \text{ a.e. } x \in \mathbb{R}$$

[This implies the second limit since one can replace f by $\tilde{f}(x) := f(-x)$.]

From calculus we know that this is true for continuous function f.

Assume we already proved the one-side Hardy-Littlewood maximal inequality.

- For any $f \in L^1(\mathbb{R})$, $\varepsilon > 0$, $\alpha > 0$, we can find (PSet 6-1, problem 2(3)) a continuous function g on $\mathbb{R}$ with compact support, s.t.

$$\int_{\mathbb{R}} |f(x) - g(x)|\, dx < \varepsilon.$$

So by the one-sided Hardy-Littlewood maximal inequality,

$$m\left(\left\{x \in \mathbb{R} : \sup_{h>0} \frac{1}{h} \int_{[x, x+h]} |f(t) - g(t)|\, dt > \alpha\right\}\right) \le \frac{\varepsilon}{\alpha}.$$

Also, by the Markov inequality,

$$m\left(\{x \in \mathbb{R} : |f(x) - g(x)| > \alpha\}\right) < \frac{\varepsilon}{\alpha}$$

Let E be the union of the two sets in the above two inequalities.

Then $m(E) < \frac{2\varepsilon}{\alpha}$, and for $\forall x \in \mathbb{R} \setminus E$, we have $(\forall h > 0)$:

$$\frac{1}{h} \int_{[x, x+h]} |f(t) - g(t)|\, dt \le \alpha, \qquad |f(x) - g(x)| \le \alpha.$$

- Now let $x \in \mathbb{R} \setminus E$. By calculus, we know that for h small enough,

$$\left| \frac{1}{h} \int_{[x, x+h]} g(t)\, dt - g(x) \right| \le \alpha.$$

So by triangle inequality, we see that for all h small enough,

$$\left| \frac{1}{h} \int_{[x, x+h]} f(t)\, dt - f(x) \right| \le 3\alpha$$

$$\Rightarrow \limsup_{h \to 0} \left| \frac{1}{h} \int_{[x, x+h]} f(t)\, dt - f(x) \right| \le 3\alpha, \quad \forall x \in \mathbb{R} \setminus E.$$

<u>Note</u>: $m(E) \le \frac{2\varepsilon}{\alpha}$.

Keeping α fixed and letting $\varepsilon \to 0$, we get

$$\limsup_{h \to 0+} \left| \frac{1}{h} \int_{[x, x+h]} f(t)\, dt - f(x) \right| \le 3\alpha, \quad \text{a.e. } x \in \mathbb{R}.$$

Letting $\alpha \to 0$, we get (trick: take $\alpha = \frac{1}{n}$).

$$\limsup_{h \to 0+} \left| \frac{1}{h} \int_{[x, x+h]} f(t)\, dt - f(x) \right| = 0, \quad \text{a.e. } x. \qquad \square$$

2. The higher dimensional case

- The general version is

Thm. (The Lebesgue differentiation theorem)

Let $f \in L^1_{loc}(\mathbb{R}^d)$. Then for a.e. $x \in \mathbb{R}^d$, one has

(1) $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(t)\, dt = f(x)$

(2) $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t) - f(x)|\, dt = 0$.

Rmk: (a) By triangle inequality, (2) $\Rightarrow$ (1).

On the other hand, one can prove (2) using (1) and a "density of $\mathbb{Q}$":

(By (1), one has $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t) - \frac{p}{q}|\, dt = |f(x) - \frac{p}{q}|$, a.e. x.
Then approximate any number by rationals ....)

(b) Let $d\mu = f\,dm$. Then (1) becomes $\frac{\mu(B(x,r))}{m(B(x,r))} \to f(x)$ a.e.

For $d=1$, this is $\frac{1}{2h} \int_{[x-h, x+h]} f\, dx \to f(x)$. Note: $\frac{1}{2h} \int_{[x-h, x+h]} f\, dx \to \left(\int_a^x f(t)\, dt \right)'$.

So in general, we may call $\lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))}$ the Lebesgue differentiation of μ (w.r.t. m at the point x).

(c) Any point x satisfying (2) is called a Lebesgue point of f.

(d) The space $L^1_{loc}(\mathbb{R}^d) = \{ f : \forall x \exists r > 0 \text{ s.t. } f \in L^1(B(x,r)) \}$.
(the space of locally integrable functions)

- The main ingredient in the proof ("Step 2" in density argument) is

Thm: (Hardy-Littlewood maximal inequality)

Suppose $f \in L^1(\mathbb{R}^d)$, and $\alpha > 0$. Then

$$m\left(\left\{ x \in \mathbb{R}^d : \sup_{r > 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)|\, dt > \alpha \right\} \right) \le \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(t)|\, dt.$$

Rmk: We will call the function

$$Mf(x) = \sup_{r > 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)|\, dt$$

the Hardy-Littlewood maximal function of f

Proof of the Lebesgue differentiation thm

(1) Almost the same as the dim = 1 case

(2) Use density of $\mathbb{Q}$ (and countability of $\mathbb{Q}$)

Left as an exercise. $\square$

To prove the Hardy-Littlewood maximal inequality, we need

<u>Lemma</u>. ("Vitali-type covering lemma") Let $\mathcal{U}=\{B(x_\alpha, r_\alpha)\}$ be a family of open balls. Suppose $U \subset \bigcup_\alpha B(x_\alpha, r_\alpha)$. Then $\forall c < m(U)$, $\exists$ finitely many pairwise disjoint balls $B_j = B(x_j, r_j)$ in the family $\mathcal{U}$, s.t. $(j=1,\dots,n)$

$$\sum_{j=1}^n m(B_j) > 3^{-d} c.$$

<u>Proof of Hardy-Littlewood maximal inequality</u>

Let $U=\{x : Mf(x) > \alpha\}$. Then $\forall x \in U$, $\exists r(x) > 0$ s.t.

$$\int_{B(x, r_x)} |f(y)| dy > \alpha \cdot m(B(x, r_x))$$

Since $U \subset \bigcup_{x\in U} B(x, r_x)$, by the lemma, one can find $x_1, \dots, x_n$ s.t. the balls $B(x_1, r_{x_1}), \dots, B(x_n, r_{x_n})$ are pairwise disjoint, and [write $B_j = B(x_j, r_{x_j})$ for simplicity]

$$\sum_{j=1}^n m(B_j) > 3^{-d} c.$$

It follows

$$c < 3^d \sum_{j=1}^n m(B_j) \le 3^d \cdot \frac{1}{\alpha} \sum_{j=1}^n \int_{B_j} |f(y)| dy \le \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| dy. \qquad \square$$

<u>Proof of "Vitali-type covering lemma"</u>.

By inner regularity of the Lebesgue, $\exists$ compact set $K \subset U$ s.t. $m(K) > c$.

Since $K \subset \bigcup_\alpha B(x_\alpha, r_\alpha)$, by compactness, $\exists B(x_1, r_1), \dots, B(x_m, r_m)$ s.t. $K \subset \bigcup_{j=1}^m B(x_j, r_j)$.

WLOG, we may assume $r_1 \ge r_2 \ge \cdots \ge r_m$.

Let $A_1 = B_1$. For each $j \ge 2$, let A_j be one of the $B(x_k, r_k)$'s, with smallest index k (i.e. largest r_k), s.t. $B(x_k, r_k) \cap \left(\bigcup_{i=1}^{j-1} A_i\right) = \phi$. [Again: we are doing some greedy choice as last lecture.]

We stop until we can not choose further A_j.

By definition, we get a finite sequence of balls $A_1, \dots, A_n$ in the family $\mathcal{U}$, which is a disjoint family.

Now for $\forall x \in B(x_k, r_k)$, we can find j s.t. $A_j \cap B(x_k, r_k) \ne \phi$. We let j be the smallest one.

Then $B(x_k, r_k) \cap (A_1 \cup \cdots \cup A_{j-1}) = \phi \Rightarrow r_k \le$ "the radius of A_j". Write $A_j = B(y_j, s_j)$.

Since $A_j \cap B(x_k, r_k) \ne \phi$, we see by triangle inequality that [choose some $x \in A_j \cap B(x_k, r_k)$]

$$B(x_k, r_k) \subset B(x, 2\cdot \text{radius}(A_j)) \subset B(y_j, 3s_j).$$

Since $K \subset \bigcup_{k=1}^m B(x_k, r_k) \subset \bigcup_{j=1}^n B(y_j, 3s_j)$, we get

$$c < m(K) \le \sum_{j=1}^n m(B(y_j, 3s_j)) = 3^d \sum_{j=1}^n m(B(y_j, s_j)). \qquad \square$$

<u>Rmk</u>: Only need to assume μ is a Radon measure that satisfies a "doubling property": $\mu(B(x, 2r)) \le C_0 \cdot \mu(B(x, r)), \forall x, r$.

• Finally we relate the "Lebesgue differentiation" with the "Radon-Nikodym derivative".

Let μ be a locally finite outer regular Borel measure on $\mathbb{R}^d$

$\mu(K)<\infty, \forall$ compact K $\quad$ $\mu(A)=\inf\{\mu(U): U\supset A, U \text{ open}\}$ $\quad$ $\mathcal{F}=\mathcal{B}$.

<u>Fact.</u> A measure $f\,dx$ is such a measure iff $f\in L^1_{loc}(\mathbb{R}^d)$.

<u>Thm:</u> Let μ be a locally finite outer regular Borel measure on $\mathbb{R}^d$.
Let $\mu=\mu_r+\mu_s$ be the decomposition s.t. $\mu_r \ll m$, $\mu_s\perp m$.
Then for a.e. $x\in\mathbb{R}^d$, one has

$$\lim_{r\to 0}\frac{\mu(B(x,r))}{m(B(x,r))}=\frac{d\mu_r}{dm}(x)$$

<u>Proof</u>: By Lebesgue differentiation theorem,

$$\lim_{r\to 0}\frac{\mu_r(B(x,r))}{m(B(x,r))}=\frac{d\mu_r}{dm}(x),\quad \text{a.e. } x\in\mathbb{R}^d.$$

So it remains to prove

$$(*)\qquad \lim_{r\to 0}\frac{\mu_s(B(x,r))}{m(B(x,r))}=0,\quad \text{a.e. } x\in\mathbb{R}^d.$$

Let $A\in\mathcal{B}$ be a Borel set in $\mathbb{R}^d$ s.t. $\mu_s(A)=0$, $m(A^c)=0$.
For $k\in\mathbb{N}$, we let

$$F_k:=\{x\in A: \limsup_{r\to 0}\frac{\mu_s(B(x,r))}{m(B(x,r))}>\frac{1}{k}\}$$

To prove $(*)$, we only need to prove $m(F_k)=0$, $\forall k$.

Since μ is outer regular, $\exists\, U_\varepsilon\supset A$ open s.t.

$$\mu_r(U_\varepsilon)+\mu_s(U_\varepsilon)=\mu(U_\varepsilon)\le\mu(A)+\varepsilon=\mu_r(A)+\varepsilon$$

So $\mu_s(U_\varepsilon)\le\varepsilon$.

But for $\forall x\in F_k$, $\exists$ open ball $B_x\subset U_\varepsilon$ s.t. $\mu_s(B_x)>\frac{1}{k}m(B_x)$.

Let $V_\varepsilon=\bigcup_{x\in F_k}B_x\subset U_\varepsilon$. Then for $\forall c<m(V_\varepsilon)$, there exists finitely many pairwise disjoint balls $B_{x_1},\dots,B_{x_n}$ s.t.

$$c<3^d\sum_{j=1}^n m(B_{x_j})\le 3^d\cdot k\sum_{j=1}^n\mu_s(B_{x_j})\le 3^d k\,\mu_s(V_\varepsilon)\le 3^d k\,\varepsilon.$$

It follows that

$$m(V_\varepsilon)\le 3^d\cdot k\,\varepsilon,\quad \forall\varepsilon>0.$$

So we conclude ~~$m(V_\varepsilon)=0$ and thus~~ $m(F_k)=0$ since $F_k\subset V_\varepsilon$, $\forall\varepsilon$. $\square$