Real Analysis

Lecture 26
06/04/2018

dim = 1 Version:
Let $f \in L^1(\mathbb{R}^1)$, $F(x) := \int_{(-\infty, x)} f(t)\,dt$.
Then $F'(x) = f(x)$ a.e. $x \in \mathbb{R}$.

Last time

- The Lebesgue differentiation theorem: Let $f \in L^1_{loc}(\mathbb{R}^d)$. Then for a.e. $x \in \mathbb{R}^d$

① $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(t)\,dt = f(x)$

LHS $= \lim_{r\to 0} \frac{\mu_f(B(x,r))}{m(B(x,r))}$

$\Uparrow$

② $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t) - f(x)|\,dt = 0.$

In other words, almost every point is a Lebesgue point

Hardy-Littlewood maximal inequality: Let $f \in L^1(\mathbb{R}^d)$, and $\alpha > 0$. Then

$\Uparrow$

$$m\left(\left\{x \in \mathbb{R}^d : \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)|\,dt > \alpha\right\}\right) \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(t)|\,dt$$

compare: Markov inequality (Lec. 9)

Vitali type covering lemma.

Let $\mathcal{U} = \{B(x_\alpha, r_\alpha)\}$ be a family of open balls that cover U.
Then $\forall\, c < m(U)$, $\exists$ disjoint balls $B(x_j, r_j)$, $1 \leq j \leq n$, s.t.
$$\sum_{j=1}^{n} m(B(x_j, r_j)) > 3^{-d} \cdot c.$$

- Vitali covering lemma (PSet 13-2): Let $\mathcal{U}$ be a Vitali covering of A:
  - ① $m(A) < \infty$
  - ② $\mathcal{U} = \{B(x_\alpha, r_\alpha)\}$.
  - ③ $\forall x \in A$, $\forall r > 0$, $\exists\, \alpha$ s.t. $x \in B(x_\alpha, r_\alpha)$, and $r_\alpha < r$

  Then $\exists$ disjoint B_i, $1 \leq i \leq N$, in $\mathcal{U}$ s.t. $\sum_{i=1}^{N} m(B_i) \geq m(A) - \varepsilon$.

$\Downarrow$

Thm A Let μ be a locally finite outer regular Borel measure on $\mathbb{R}^d$.
Let $\mu = \mu_r + \mu_s$ be the Lebesgue decomposition of μ w.r.t. m, i.e. $\mu_r \ll m$, $\mu_s \perp \mu$.
Then for a.e. $x \in \mathbb{R}^d$, one has $\lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = \frac{d\mu_r}{dm}(x)$.

"Lebesgue differentiation" $\Leftrightarrow$ "Radon-Nikodym derivative".

As a consequence, we see
$$\mu_r(A) = \int_A \begin{pmatrix}\text{the Lebesgue diff.}\\ \text{of } \mu \text{ w.r.t. } m\end{pmatrix} dm$$

As another corollary,
$$\mu \perp m \Leftrightarrow \lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = 0, \text{ a.e. } x$$

Rmk. One can replace $B(x,r)$'s by a family of sets $E_r(x)$'s that "shrinks nicely" to x, i.e. ① $E_r(x) \subset B(x,r)$

② $\exists\, \alpha$ s.t. $m(E_r(x)) > \alpha\, m(B(x,r))$.

and conclude that
$$\lim_{r \to 0} \frac{\mu(E_r(x))}{m(E_r(x))} = f(x), \text{ a.e. } x$$

- Another simple consequence of Lebesgue differentiation theorem:

Let A be a Lebesgue measurable set.
The density of A at a point x is
$$D_A(x) := \lim_{r \to 0} \frac{m(A \cap B(x,r))}{m(B(x,r))}.$$

Fact: $D_A(x) = \begin{cases} 1, & \text{a.e. } x \in A \\ 0, & \text{a.e. } x \in A^c. \end{cases}$

Reason: Take $f = \chi_A$ in Lebesgue Differentiation theorem.

Today: The fundamental theorem of Calculus

1. The case of monotone functions

- Let's start with a non-negative integrable function f defined on $\mathbb{R}$.

Using f we can define a Borel measure $\mu_f = f\,dm$ s.t. $\mu_f([a,b]) = \int_{[a,b]} f(t)\,dt$.

We have seen last time that if we let

$$F(x) = \int_{(-\infty, x]} f(t)\,dt = \mu_f((-\infty, x]).$$

Then $F'(x) = f(x)$ for a.e. $x \in \mathbb{R}$.

Note: It is possible that F is not differentiable at some points.

e.g., let $f(x) = \chi_{[0,1]}$. Then $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x \le 1, \\ 1, & x > 1. \end{cases} \rightsquigarrow F' = \chi_{[0,1]}$ a.e.

So F is NOT differentiable at $x=0$ and $x=1$.

~~Since $f \ge 0$, the function F is always monotone increasing.~~

- If one take μ to be singular measures, one can even make the function $F(x) = \mu((-\infty, x])$ a discontinuous function. e.g.

Let $\mu = \delta_0 + \delta_1 + \chi_{[0,1]}dm \Rightarrow F(x) = \begin{cases} 0, & x < 0 \\ 1+x, & 0 \le x < 1 \\ 3, & 1 \le x < \infty. \end{cases}$

In this case we still have $F'(x) = \chi_{[0,1]}$ a.e.

Note: Although $F(x) = \mu((-\infty, x])$ could be non-differentiable, or even discontinuous, it is always monotone increasing and right-continuous.

$$a < b \Rightarrow F(b) - F(a) = \mu((a,b]) \ge 0$$

Let $x_n \searrow x_0$. Then $(-\infty, x_n] \to (-\infty, x_0]$. By monotone convergence thm, $F(x_n) = \mu((-\infty, x_n]) \to F(x_0) = \mu((-\infty, x_0])$.

Think: Why we can't use this method to prove "left-continuity"?

Conversely, in PSet 9-2, problem 2 we have seen that for any monotone increasing right continuous function $F: \mathbb{R} \to \mathbb{R}$, one can define a metric outer measure μ_F (and thus a Borel measure) on $\mathbb{R}$ s.t. $\mu_F((a,b]) = F(b) - F(a)$.

It is regular By PSet 10-1

In view of the close relation between (regular) Borel measures and monotone increasing right continuous functions, the following theorem is natural:

Thm (Monotone Differentiation Theorem)

Any monotone function $F: \mathbb{R} \to \mathbb{R}$ is a.e. differentiable.

Proof: WLOG, we may assume F is monotone increasing. [otherwise we can consider $-F$.]

- Observation 1. The set of points where F is discontinuous is countable.

Reason. Let $\{x_\alpha\}$ be the set of discontinuity. Then the intervals $\{(F(x_\alpha -), F(x_\alpha +))\}$ are disjoint and non-empty in $\mathbb{R}$. One can only have at most countably many x_α. (PSet 2-1, Problem 2)

• As a consequence, if we let $G(x) = F(x+)$. Then $G = F$ a.e.

Note: By def., G is right continuous, and $G(x+h) - G(x) = \begin{cases} \mu_G((x, x+h]), & h>0 \\ \mu_G((x+h, x]), & h<0 \end{cases}$

Applying Thm A to the measure μ_G (with $B(x,r)$ replaced by the family $\{(x-r, x]\}$ and $\{(x, x+r]\}$), we get

$$\lim_{h\to 0+} \frac{G(x+h) - G(x)}{h} = \lim_{h\to 0+} \frac{\mu_G((x, x+h])}{m((x, x+h])}$$

and $$\lim_{h\to 0-} \frac{G(x+h) - G(x)}{h} = \lim_{h\to 0-} \frac{\mu_G((x+h, x])}{m((x+h, x])}$$

exists a.e. and are equal a.e. In other words, the function G is differentiable a.e.

• It remains to prove that if we let $H = G - F$, then $H' = 0$ a.e.

Let $\{x_n\}$ be the set of the points of discontinuity of F, i.e. be s.t. $H(x_n) \neq 0$.

By monotonicity, $H(x_n) > 0$ for all n. Moreover,

$$\sum_{j: |x_j| < N} H(x_j) \le F(N) - F(-N) < \infty.$$

check this!

So if we let $\nu = \sum_j H(x_j) \delta_{x_j}$, then ν is a locally finite outer regular Borel measure on $\mathbb{R}$. Moreover, by definition we have $\nu \perp m$.

So according to Thm A, for a.e. x we have

$$\left|\frac{H(x+h) - H(x)}{h}\right| \le \frac{H(x+h) + H(x)}{h} \le \frac{\nu((x-2|h|, x+2|h|))}{h} \to 0.$$

So we conclude that $H' = 0$ a.e. x, i.e. $F' = G'$ a.e. x. $\square$

2. Functions of bounded variation

• What if we started with a function $f \in L^1(\mathbb{R})$, instead of with nonnegative integrable f?

Then the function $F(x) := \mu_f((-\infty, x])$ is no longer monotone increasing.

Rough idea:

"Non-negative" theory	"Absolute integrable" theory
$f \ge 0$, $f \in L^1$	$f \in L^1$: $f = f_1 - f_2$, (f_i non-negative)
measure μ	signed measure $\mu = \mu_1 - \mu_2$, (μ_i measures)
"monotone increasing, (bounded)"	"differences of monotone increasing functions?" ??

• Now let $F: \mathbb{R} \to \mathbb{R}$ be a bounded monotone increasing function, say $|F(x)| < C$, $\forall x$. ~~We have seen that~~ Then of course for any $x_1 < x_2 < \cdots < x_n$, one has

$$\sum_{i=1}^{n} |F(x_{i+1}) - F(x_i)| < |F(+\infty)| + |F(-\infty)| < 2C.$$

If $F = F_1 - F_2$, where F is no longer monotone, but F_1, F_2 are monotone and $|F_i(x)| < C_i$,

Then by triangle inequality, $\sum_{i=1}^{n} |F(x_{i+1}) - F(x_i)| < 4C$ for $\forall x_1 < x_2 < \cdots < x_n$.

· **Def.** Let $F:\mathbb{R}\to\mathbb{R}$ be a function. We define the <u>total variation</u> of F to be
$$\|F\|_{TV} := \sup\left\{ \sum_{i=1}^{n-1} |F(x_{i+1})-F(x_i)| : \forall n,\ \forall x_1<x_2<\cdots<x_n \right\}$$
We say F has <u>bounded variation</u> if $\|F\|_{TV} < +\infty$.

Rmk. • Given any interval $[a,b]$, one can also define
$$\|F\|_{TV([a,b])} := \sup\left\{ \sum_{i=1}^{n-1} |F(x_{i+1})-F(x_i)| : \forall n,\ \forall a\le x_1<x_2<\cdots<x_n\le b \right\}.$$
and then define functions of bounded variation on $[a,b]$.

• One can check that ① $\|F+G\|_{TV} \le \|F\|_{TV}+\|G\|_{TV}$ (Exercise) ← Also one has $\|F\|_{TV([a,b])}+\|F\|_{TV([b,c])} = \|F\|_{TV([a,c])}$.

② $\|cF\|_{TV} = |c|\cdot\|F\|_{TV}$.

③ $\|F\|_{TV}=0 \Leftrightarrow F$ is a constant.

So $\|\cdot\|_{TV}$ is a norm on the space
$$\{F : \|F\|_{BV} < +\infty\}/\sim, \quad \text{where } F\sim G \Leftrightarrow F-G=\text{constant}.$$

• The space of functions of bounded variation is denoted by BV.
It is easy to see that BV is a vector space.

• If F is of bounded variation, then F is bounded.

• By definition it is easy to see that any bounded increasing function is of bounded variation.
As a consequence, the difference of two bounded increasing function is of bounded variation.
Conversely, we have

Thm. A function $F:\mathbb{R}\to\mathbb{R}$ is of bounded variation iff F is the difference of two bounded increasing functions.

Proof. [<u>Motivation</u>: $f(x)=f^+(x)-f^-(x)$, where $f^+(x)=\begin{cases} f(x), & f(x)\ge 0\\ 0, & f(x)<0\end{cases}$, $f^-(x)=\begin{cases} -f(x), & f(x)<0\\ 0, & f(x)\ge 0.\end{cases}$]

Define the "positive variation function" of F to be
$$F^+(x) := \sup\left\{ \sum_{i=1}^{n-1} \max(F(x_{i+1})-F(x_i), 0)\ \forall n,\ \forall x_1<x_2<\cdots<x_n\le x \right\}$$
By def, F^+ is a monotone increasing function, and $0\le F^+(x)\le \|F\|_{TV}$.
Let $F_- = F^+ - F$. Then $F = F^+ - F_-$.

<u>Claim</u>: F_- is monotone increasing. (F_- is bounded since both F and F^+ are).

To see this we suppose $a<b$. We want to show $F_-(a)\le F_-(b)$, i.e.
$$F^+(b) \ge F^+(a) + F(b) - F(a). \quad (*)$$
If $F(b)\le F(a)$, then by monotonicity of F^+ we get $(*)$.

Now we suppose $F(b) > F(a)$. Then any partition

$$x_1 < x_2 < \cdots < x_n \le a$$

give rise to a partition

$$x_1 < x_2 < \cdots < x_n < x_{n+1} = a < x_{n+2} = b \quad \text{or} \quad x_1 < x_2 < \cdots < x_n = a < x_{n+1} = b.$$

So by definition of F^+, we must have

$$F^+(b) \ge F^+(a) + F(b) - F(a).$$

This completes the proof. $\square$

- There are many corollaries.

Cor 1: If F is of bounded variation, then F is differentiable a.e.

Cor 2: There exists continuous functions that is bounded but NOT of bounded variation on any interval.

[Just take a continuous function that is NOT differentiable anywhere]

Cor 3. If F is of bounded variation, then F is discontinuous at at most countably many points.

Cor 4: If F is of bounded variation, and $G(x) = F(x+)$. Then $F' = G'$ a.e. $x \in \mathbb{R}$.

Note: It is enough to assume F is locally bounded variation.

3. The fundamental theorem of calculus

- In calculus we learned two versions of the fundamental theorem of calculus.

Version 1: If $f : [a,b] \to \mathbb{R}$ is continuous, then $F(x) = \int_a^x f(t)\,dt$ is differentiable and $F'(x) = f(x)$, $\forall x \in (a,b)$.

Version 2: If F is differentiable, and $F'(x)$ is Riemann integrable on $[a,b]$, then $\int_a^x F'(t)\,dt = F(x) - F(a)$.

We have seen last time that version 1 holds for $f \in L^1(\mathbb{R})$ (and we use Lebesgue integral, replace = by "= a.e.").

Now we would like to extend version 2 to more general functions.

- It would be nice if version 2 holds as long as F' exists a.e. e.g. for any BV functions.

Unfortunately this is NOT true:

If we take F to be the Cantor function. (P24. You are supposed to read this in PSet 1-1)

Then F is increasing, and $F' = 0$ a.e. x.

So we have $\int_0^1 F'(t)\,dt = 0 < F(1) - F(0) = 1$.

In other words the version 2 fails even for monotone functions.

- So we need extra assumptions on F to get

$$F(x) = F(a) + \int_{[a,x]} F'(t)\,dt.$$

Suppose this is true. Then according to the absolute continuity of Lebesgue integral, (see Lec. 11)
for any $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $A = (a_1, b_1) \cup \cdots \cup (a_n, b_n)$,
and $m(A) < \delta$, then $\sum_{j=1}^{n} |F(b_j) - F(a_j)| < \int_A |F'(t)|\,dt < \varepsilon$.

This motivates the following definition (which is a necessary condition for "Version 2")

Def. A function $F : \mathbb{R} \to \mathbb{R}$ is <u>absolutely continuous</u> if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for any disjoint intervals $[a_1, b_1], \cdots, [a_n, b_n]$ satisfying $\sum_{j=1}^{n} (b_j - a_j) < \delta$, one has

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| < \varepsilon.$$

Prop 1. If F is absolutely continuous on $[a, b]$, then F has bounded variation on $[a, b]$.

Proof. By def., $\exists \delta > 0$ s.t. $\sum_{j=1}^{n} (b_j - a_j) < \delta \Rightarrow \sum_{j=1}^{n} |F(b_j) - F(a_j)| < 1$.

Let $a = x_1 < x_2 < \cdots < x_n = b$ be any partition s.t. $x_k - x_{k-1} < \delta$, where $n = \left[\frac{b-a}{\delta}\right] + 2$.

Then $\|F\|_{TV} = \sum_{i=1}^{n-1} \|F\|_{TV([x_i, x_{i+1}])} < n$. $\square$

Prop 2. If F is absolutely continuous on $[a, b]$, and $F'(x) = 0$ a.e. x, then F is a constant.

Proof. It suffices to show $F(a) = F(b)$, since we can replace $[a, b]$ by any subinterval.

Let $A = \{x \in (a, b) : F'(x) = 0\}$. Then $\forall x \in A$,

$$\lim_{h \to 0} \left| \frac{F(x+h) - F(x)}{h} \right| = 0.$$

Fix ε. Then $\forall \eta > 0$, $\exists x \in (a_x^\eta, b_x^\eta) \subset (a, b)$ s.t. $|F(b_x^\eta) - F(a_x^\eta)| \le \varepsilon\,(b_x^\eta - a_x^\eta) < \varepsilon\eta$

The family of intervals $\{(a_x^\eta, b_x^\eta)\}$ form a Vitali covering of A.

One can find finite subcollection $(a_1, b_1), \cdots, (a_n, b_n)$, disjoint, s.t.

$$\sum_{i=1}^{n} (b_i - a_i) \ge m(A) - \delta = (b - a) - \delta.$$

On the other hand,

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| \le \varepsilon \cdot \sum_{i=1}^{n} (b_i - a_i) \le \varepsilon (b - a)$$

<u>Observation</u>. The complement of $\bigcup_{i=1}^{n} (a_i, b_i) \subset [a, b]$ is a finite union of closed intervals, $\bigcup_{k=1}^{m} [\alpha_k, \beta_k]$, of length ~~total~~ no more than δ.

So by absolute continuity, $\sum_{j=1}^{m} |F(\beta_j) - F(\alpha_j)| < \varepsilon$.

$$\Rightarrow \quad F(b) - F(a) \le \sum_i |F(b_i) - F(a_i)| + \sum_j |F(\beta_j) - F(\alpha_j)| \le \varepsilon (b - a) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get $F(b) = F(a)$. $\square$

Now let's turn to the fundamental theorem of calculus, version 2. We first prove

Thm. (The fundamental theorem of calculus for monotone functions)
Let F be any monotone increasing function (so F' exists a.e.). Then

$$\int_{[a,b]} F'(x)\,dx \leq F(b) - F(a).$$

In particular, if F is bounded, then $F' \in L^1(\mathbb{R})$.

Proof: Let $f_n(x) = \dfrac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}}$. By monotone differentiation theorem, $f_n(x) \to F'(x)$ a.e. x.
($\Rightarrow F'$ is measurable, non-negative).

So Fatou's lemma implies

$$\int_{[a,b]} F'(x)\,dx \leq \liminf_{n\to\infty} \int_{[a,b]} f_n(x)\,dx$$

But

$$\begin{aligned}\int_{[a,b]} f_n(x)\,dx &= n\int_{[a,b]} F(x+\tfrac{1}{n})\,dx - n\cdot\int_{[a,b]} F(x)\,dx \\ &= n\int_{[b,b+\frac{1}{n}]} F(x)\,dx - n\int_{[a,a+\frac{1}{n}]} F(x)\,dx \\ &\to F(b) - F(a).\end{aligned}$$

So we get $\int_{[a,b]} F'(x)\,dx \leq F(b) - F(a)$. □

Rmk: As we have seen, we can't have "=" in general.

Cor.: Let F be a BV function. Then $F' \in L^1$.

Finally we arrive at

Thm. (The fundamental theorem of calculus (version 2))
Let $F : [a,b] \to \mathbb{R}$ be absolutely continuous. Then $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Proof: We have seen that F is AC $\Rightarrow$ F is BV $\Rightarrow$ F' exists a.e. and $F' \in L^1$.
Let $G(x) = \int_a^x F'(t)\,dt$. Then by absolute continuity of Lebesgue integrals, G is AC.
It follows that $F - G$ is AC.
By Lebesgue differentiation theorem (i.e. version 1 of the fundamental theorem of calculus),
we have $F' = G'$ a.e. x, i.e. $(F-G)' = 0$ a.e.
So by proposition 2, $F - G$ is a constant.
Since ~~$F(a) = G(a)$~~ $F(a) - G(a) = F(a)$, we must have ~~$F(x) = G(x)$, $\forall x$~~ $F(b) - G(b) = F(a)$. □

Rmk. Conversely if $F' \in L^1$ and $F(x) = F(a) + \int_a^x F'(t)\,dt$, then we have already seen that F is AC.
So AC $\Leftrightarrow$ the fundamental theorem of calculus holds!