Real Analysis

Lecture 27
06/07/2018

Last time.

- BV functions: $\|f\|_{TV} = \sup\left\{ \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| : \forall n, \forall x_1 < x_2 < \cdots < x_n \right\} < \infty$.
  - f is BV function $\Leftrightarrow$ $f = f_1 - f_2$, both f_1, f_2 are bounded increasing function
- AC functions: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall$ disjoint $(a_1, b_1), \cdots, (a_n, b_n)$, $\sum_{i=1}^n (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon$.
  - AC function on $[a, b]$ is BV.
- Monotone differentiation theorem: Any monotone increasing function is a.e. differentiable, and $f' \in L^1$,

$$\int_{[a,b]} f'(t)\, dt \le f(b) - f(a).$$

$\Rightarrow$ Any $\overbrace{\ }^{\text{locally}}$ BV function is a.e. differentiable, and $f' \in L^1$!

- The fundamental theorem of calculus

  Version 1: If $f \in L^1(\mathbb{R})$, $F(x) = \int_{(-\infty, x)} f(t)\, dt$, then $F' = f$ a.e.

  Version 2. If $F : [a, b] \to \mathbb{R}$ is AC, then $\int_{[a,b]} F'(t)\, dt = F(b) - F(a)$.

  $(\Leftarrow)$

Today: The Rademacher differentiation theorem

1. Some applications of the fundamental theorem of calculus

- Integration by parts

Thm. (Integration by parts formula) Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous. Then

$$\int_{[a,b]} f(t) g'(t)\, dt = f(b) g(b) - f(a) g(a) - \int_{[a,b]} f'(t) g(t)\, dt$$

Proof. According to PSet 14-1, problem 3, we know fg is absolutely continuous. So we can integrate the identity

$$fg' = (fg)' - f'g \quad \text{a.e.}$$

to get

$$\int_{[a,b]} f(t) g'(t)\, dt = \int_{[a,b]} (fg)'\, dt - \int_{[a,b]} f'(t) g(t)\, dt$$

$$= f(b) g(b) - f(a) g(a) - \int_{[a,b]} f'(t) g(t)\, dt. \quad \square$$

- AC functions as BV functions:

Thm. Let f be an absolutely continuous function on $[a, b]$. Then

$$\|f\|_{TV([a,b])} = \int_{[a,b]} |f'(t)|\, dt$$

Proof. For any partition $a = x_1 < x_2 < \cdots < x_n = b$, we have

$$\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| = \sum_{i=1}^{n-1} \left| \int_{[x_i, x_{i+1}]} f'(t)\, dt \right| \le \sum_{i=1}^{n-1} \int_{[x_i, x_{i+1}]} |f'(t)|\, dt = \int_{[a,b]} |f'(t)|\, dt.$$

It follows that $\|f\|_{TV([a,b])} \leq \int_{[a,b]} |f'(t)|\,dt$.

- To prove the converse, we give ourselves an ε-room.

Since $f' \in L^1$, we can find (by PSet 6-1-2) a step function g on $[a,b]$ s.t. $h := f' - g$ satisfies $\|h\|_{L^1} < \varepsilon$

We set $G(x) = \int_{[a,x]} g(t)\,dt$, $H(x) = \int_{[a,x]} h(t)\,dt$. Then

$$f(x) = f(a) + \int_{[a,x]} f'(t)\,dt = f(a) + G(x) + H(x).$$

Since $H' = h \in L^1$, H is AC, so by what we just proved, $\|H\|_{TV([a,b])} \leq \|h\|_{L^1} < \varepsilon$.

So by triangle inequality,

$$\|f\|_{TV([a,b])} \geq \|G\|_{TV([a,b])} - \|H\|_{TV([a,b])} \geq \|G\|_{TV([a,b])} - \varepsilon.$$

Since g is a step function, we can partition $[a,b]$ into

$$a = x_1 < x_2 < \cdots < x_n = b$$

and write $g = \sum_{i=1}^{n-1} c_i \chi_{[x_i, x_{i+1}]}$ It follows

$$\|G\|_{TV([a,b])} \geq \sum_{i=1}^{n-1} |G(x_{i+1}) - G(x_i)| = \sum_{i=1}^{n-1} \left| \int_{[x_i, x_{i+1}]} g(t)\,dt \right|$$

$$= \sum_{i=1}^{n-1} \int_{[x_i, x_{i+1}]} |g(t)|\,dt = \int_{[a,b]} |g(t)|\,dt = \|g\|_{L^1([a,b])}$$

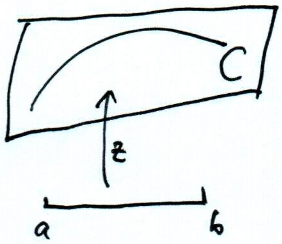
$$\geq \|f'\|_{L^1([a,b])} - \varepsilon.$$

So we get $\|f\|_{TV([a,b])} \geq \|f'\|_{L^1([a,b])} - 2\varepsilon$, $\forall \varepsilon$.

This implies $\|f\|_{TV([a,b])} = \|f'\|_{L^1([a,b])}$. $\square$

- Rmk 1: The theorem holds for complexed-valued functions.

Rmk 2: A geometric application: length of rectifiable curves.



- A curve $z : [a,b] \to \mathbb{R}^2$ is rectifiable if $\exists\, C < +\infty$ s.t. $\forall a = t_0 < t_1 < \cdots < t_N = b$, one has
$$\sum_{j=1}^{N} |z(t_j) - z(t_{j-1})| \leq M.$$
- The length of a rectifiable curve C is
$$L(C) = \sup \sum_{j=1}^{N} |z(t_j) - z(t_{j-1})|.$$

Using our language, $\cdot$ C is rectifiable $\Leftrightarrow$ z is BV.

$\cdot$ $L(C) = \|z\|_{TV}$.

Cor: Let $z(t) = (x(t), y(t)) : [a,b] \to \mathbb{R}^2$ be a planar curve.
If x, y are AC functions, then the curve is rectifiable, and $L(C) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

2. Lipschitz continuous functions

- Lipschitz continuity is a conception "between" continuous and differentiable (with bounded derivative)

<u>Def.</u> A function $f: \mathbb{R} \to \mathbb{R}$ is <u>Lipschitz continuous</u> if $\exists C > 0$ s.t.

$$|f(x) - f(y)| \le C|x-y|, \quad \forall x, y.$$

More generally, a function $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ is <u>Lipschitz continuous</u> if $\exists C > 0$ s.t.

$$|f(x) - f(y)| \le C|x-y|, \quad \forall x, y \in A.$$

<u>Notation</u>: $Lip(f) = \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|} : x \neq y \right\}$.

- <u>Example/Prop.</u>: If $f: \mathbb{R} \to \mathbb{R}$ is AC and $|f'| \le M$ a.e., then f is Lipschitz continuous, and $Lip(f) \le M$.

<u>Proof.</u> $|f(x) - f(y)| = \left| \int_{[x,y]} f'(t)\, dt \right| \le \int_{[x,y]} |f'(t)|\, dt \le M|x-y|$. $\square$

Conversely, we have

<u>Prop.</u>: Let $F: [a,b] \to \mathbb{R}$ be Lipschitz continuous with $Lip(F) = M$. Then $\exists f \in L^\infty([a,b])$ $(\subset L^1([a,b]))$ with $\|f\|_{L^\infty} \le M$ s.t.

$$F(x) = F(a) + \int_{[a,x]} f(t)\, dt.$$

<u>Proof</u>: - We first define a linear functional on the space of step functions via

$$L\left(\sum_{i=1}^n a_i \chi_{[x_i, x_{i+1}]} \right) := \sum_{i=1}^n a_i (F(x_{i+1}) - F(x_i)). \quad \leftarrow \text{(check linearity!)}$$

Since F is Lipschitz, we have, for any step function $h_n = \sum a_i \chi_{[x_i, x_{i+1}]}$

$$|L(h)| \le \sum_{i=1}^n |a_i| \cdot M \cdot |x_{i+1} - x_i| = M \|h\|_{L^1}. \quad (*)$$

- Since any function in $L^1([a,b])$ can be approximated in L^1-norm by step functions, from $(*)$ we can define, for $g = \lim_{n\to\infty} h_n$, where $g \in L^1$ and h_n are step functions,

$$L(g) = \lim_{n\to\infty} L(h_n). \quad \text{[Check: why well-defined, why linear]}$$

Moreover, $L(g) \le M \|g\|_{L^1}$, $\forall g \in L^1$.

- So L is a continuous linear functional on L^1. By Riesz representation theorem, $\exists f \in L^\infty([a,b])$ s.t. $L(g) = \int_{[a,b]} f g\, dt$, and $\|f\|_{L^\infty} \le M$.

In particular, if we take $g = \chi_{[a,x]}$, we get

$$F(x) - F(a) = L(g) = \int_{[a,b]} f \cdot \chi_{[a,x]}\, dt = \int_{[a,x]} f(t)\, dt. \quad \square$$

Cor.: If $f: \mathbb{R} \to \mathbb{R}$ is Lipschitz with $Lip(f) = M$, then f is AC, and $|f'| \le M$ a.e.

Cor.: If $f: \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, then f is a.e. differentiable

- Now consider functions of d-variables, i.e. $f: \mathbb{R}^d \to \mathbb{R}$

As in calculus, the <u>directional derivative</u> of f in the direction $v \in \mathbb{R}^d$ at $x_0 \in \mathbb{R}^d$ is

$$D_v f(x_0) = \lim_{\mathbb{R} \ni h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}.$$

We say f is <u>differentiable</u> at $x_0 \in \mathbb{R}^d$ if there is a linear mapping $L: \mathbb{R}^d \to \mathbb{R}$ s.t.

$$\lim_{\mathbb{R}^d \ni h \to 0} \frac{|f(x_0+h) - f(x_0) - L(h)|}{|h|} = 0. \qquad (\text{"}L = df_{x_0}\text{"})$$

Note: • If f is differentiable, then $D_v f(x_0) = L(v)$.

In this case, if we write $\partial_i f = L(e_i)$, where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ slot.

then $D_v f(x_0) = \langle \partial_1 f, \cdots, \partial_d f \rangle \cdot v$

• However, it is possible that $D_v f$ exists for all v, but f is not differentiable

[In general, one need "continuous partial derivatives" to get "differentiability"]

- The main theorem we want to prove is

Thm: (Rademacher differentiation theorem) Let $f: \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous. Then f is differentiable at a.e. $x \in \mathbb{R}^d$

Proof. We have proved the theorem for $d=1$.

We will inductively prove the theorem, using Fubini's theorem.

We first prove the existence of directional derivatives.

~~Step 1. The existence of directional derivatives.~~

• Since f is continuous, $D_v f(x_0)$ exists if and only if

$$\limsup_{\mathbb{Q} \ni h \to 0} \frac{f(x_0+hv) - f(x_0)}{h} = \liminf_{\mathbb{Q} \ni h \to 0} \frac{f(x_0+hv) - f(x_0)}{h}.$$

It follows that for $\forall v \in \mathbb{R}^d$, the set

$$E_v := \{x_0 \in \mathbb{R}^d : D_v f(x_0) \text{ does not exist}\}$$

is a Borel set, and thus a Lebesgue measurable set.

Moreover, the function $D_v f$ is a measurable function on E_v^c.

Since f is Lipschitz, we see $D_v f$ is bounded.

- We claim: $m(E_v)=0$, $\forall v$.

Obviously this is true for $v=0$.

For $v\neq 0$, by applying a linear transformation, we may assume $v=e_1$.

So we need to show $\partial_1 f = \frac{\partial f}{\partial x_1}$ exists a.e.

We split $\mathbb{R}^d = \mathbb{R}\times\mathbb{R}^{d-1}$. Fix any $(x^0, y^0)\in\mathbb{R}\times\mathbb{R}^{d-1}$. By def,

$\partial_1 f(x^0, y^0)$ exists $\Leftrightarrow$ The function $x\mapsto f(x,y^0)$ is differentiable at x^0.

But this function is also Lipschitz. So by the result of dimension 1, the set

$$E^{y_0} = \{x^0\in\mathbb{R} : (x^0,y^0)\in E_v\}$$ is a null set.

So by Tonelli's thm,

$$m(E_v) = \iint_{\mathbb{R}\times\mathbb{R}^{d-1}} \chi_{E_v}\, dx dy = \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} \chi_{E^{y_0}}(x)\, dx\, dy = 0.$$

- As an immediate consequence, we see

$$m\Big(\bigcup_{v\in\mathbb{Q}^d} E_v\Big) = 0.$$

In other words, for a.e. $x_0\in\mathbb{R}^d$, $D_v f$ exists for all $v\in\mathbb{Q}^d$.

- Now let $g:\mathbb{R}^d\to\mathbb{R}$ be any ~~Lipschitz continuous~~ smooth function which is compactly supported i.e. $g\in C_0^\infty(\mathbb{R}^d)$

Since $D_v f$ is bounded, g is bounded, we apply DCT to get

$$\begin{aligned}
\int_{\mathbb{R}^d} D_v f(x)\, g(x)\, dx &= \int_{\mathbb{R}^d} \lim_{\mathbb{R}\ni h\to 0} \frac{f(x+hv)-f(x)}{h}\, g(x)\, dx \\
&= \lim_{\mathbb{R}\ni h\to 0} \int_{\mathbb{R}^d} \frac{f(x+hv)-f(x)}{h}\, g(x)\, dx \\
&= \lim_{\mathbb{R}\ni h\to 0} \int_{\mathbb{R}^d} f(x)\frac{g(x-hv)-g(x)}{h}\, dx \\
&= \int_{\mathbb{R}^d} f(x)\, D_{-v} g(x)\, dx. \\
&= \int_{\mathbb{R}^d} f(x)\cdot \langle \partial_1 g, \cdots, \partial_d g\rangle\cdot(-v)\, dx
\end{aligned}$$

$\leftarrow$ g compactly supported $\Rightarrow$ the integrand is compactly supported, with common support for $h<1$.

Since the RHS is linear in v, we conclude that the LHS is also linear in v.

So if $v = \sum v_i e_i$, then

$$\int_{\mathbb{R}^d} (D_v f)(x)\, g(x)\, dx = \sum_{i=1}^d v_i \int_{\mathbb{R}^d} (\partial_i f)(x)\, g(x)\, dx.$$

i.e.

$$\int_{\mathbb{R}^d} (D_v f - v\cdot\nabla f)\, g(x)\, dx = 0, \quad \forall\, g \text{ smooth and compactly supported}.$$

(where $\nabla f = (\partial_1 f, \cdots, \partial_d f)$)

• We need

Lemma: If $h \in L^1_{loc}(\mathbb{R}^d)$, and $\int_{\mathbb{R}^d} h(x)\varphi(x)dx = 0$, $\forall \varphi \in C_0^\infty(\mathbb{R}^d)$, then $h = 0$ a.e.

Proof: Suppose $h > 0$ on a set of positive measure.
Then $\exists$ compact set K s.t. $m(K) > 0$, and $\exists \varepsilon > 0$ s.t. $f \geq \varepsilon$ on K.
Let $U_1 \supset U_2 \supset \cdots \supset K$ be a decreasing sequence of open sets s.t. $\bigcap U_i = K$.
Take $\varphi_i \in C_0^\infty(U_i)$ s.t. $0 \leq \varphi_i \leq 1$, $\varphi_i = 1$ on K.

(In Lec. 19 we showed there exist continuous functions. Apply convolution to get C_0^∞ functions)

Then
$$0 = \int_{\mathbb{R}^d} h(x)\varphi_i(x)dx \geq \varepsilon \cdot m(K) - \int_{U_i \setminus K} |f(x)|dx \to \varepsilon\, m(K).$$ contradiction $\square$

It follows that
$$(*) \quad D_v f(x_0) = v \cdot \nabla f(x_0), \quad \text{a.e. } x_0 \in \mathbb{R}^d, \ \forall v \in \mathbb{Q}^d.$$

• Now let x_0 be s.t. $(*)$ holds for all $v \in \mathbb{Q}^d$, i.e. $x_0 \in A = \{x \in \mathbb{R}^d : D_v f(x) = v \cdot \nabla f(x), \forall v \in \mathbb{Q}^d\}$.
We prove f is differentiable at x_0. ($\Rightarrow$ the theorem holds)

We define $F: \mathbb{R}^d \to \mathbb{R}^d$ by
$$F(h) = f(x_0 + h) - f(x_0) - h \cdot \nabla f(x_0)$$

Note: • $F(0) = 0$
• F is Lipschitz
• $D_v F(0) = 0$ (by $(*)$), $\forall v \in \mathbb{Q}^d$.

Claim: $\displaystyle\lim_{\mathbb{R}^d \ni h \to 0} \frac{|F(h)|}{|h|} = 0.$

Let $\varepsilon > 0$, $h \in \mathbb{R}^d$. We write $h = ru$, where $r = |h|$, $u \in S^{d-1}$
Take $v \in \mathbb{Q}^d \cap S^{d-1}$ s.t. $|u - v| < \varepsilon$. [We can choose v from a finite subset $V_\varepsilon \subset \mathbb{Q}^d$ that depends on ε]

Since $D_v F(0) = 0$, $\forall v \in V_\varepsilon$, we get
$$\frac{|F(rv) - F(0)|}{r} \leq \varepsilon, \quad \forall v \in V_\varepsilon$$
$$\Rightarrow |F(rv)| \leq \varepsilon r, \quad \forall v \in V_\varepsilon$$
But since F is Lipschitz,
$$|F(h) - F(rv)| = |F(ru) - F(rv)| \leq \mathrm{Lip}(F) \cdot r|u - v| \leq \mathrm{Lip}(F) \cdot r\varepsilon.$$
So we get $|F(h)| \leq (\mathrm{Lip}(F) + 1) r\varepsilon = (\mathrm{Lip}(F) + 1) \cdot |h| \varepsilon$.
Letting $\varepsilon \to 0$, we get the claim. $\square$.

Rmks: • One can easily extends the theorem to locally Lipschitz maps $F: \mathbb{R}^d \to \mathbb{R}^m$.

• Stepanov: Any measurable $f: \mathbb{R}^d \to \mathbb{R}$ is differentiable a.e. in $S(f) := \{x \in \mathbb{R}^d : \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} < \infty\}$

• $\forall A \subset \mathbb{R}$ with $m(A) = 0$, $\exists$ Lipschitz function $f: \mathbb{R} \to \mathbb{R}$ s.t. f is not differentiable at $\forall x \in A$.

1990 $\to$ • $\exists A \subset \mathbb{R}^d$ $(d \geq 2)$ with $m(A) = 0$, s.t. $\forall$ Lipschitz function $f: \mathbb{R}^d \to \mathbb{R}$ is differentiable at at least one point in A.

2010 $\to$ • $\forall A \subset \mathbb{R}^2$ with $m(A) = 0$, $\exists$ Lipschitz function $f: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. f is not differentiable at $\forall x \in A$.

• same for $f: \mathbb{R}^d \to \mathbb{R}^m$, $m \geq d$.

2015 $\to$ • For $m < d$: $\exists\, m(A) = 0$ s.t. $\forall$ Lipschitz function $f: \mathbb{R}^d \to \mathbb{R}^m$ is differentiable at at least one point in A.