<u>Real Analysis</u>

Lecture 28
06/14/2018

Course Review

Final Exam
06/25, 8:30-11:30
@ 5403, 5404

In this course, we mainly learned
- Measure theory
- Integration theory
- The fundamental theorem of calculus

Of course, everything start with

- $\boxed{\text{Measure}}$ = A triple $(X, \mathcal{F}, \mu)$, where
  - X is a set.
  - $\mathcal{F}$ is a <u>σ-algebra</u>, i.e. $\phi \in \mathcal{F}$; $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$; $A_i \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.
  - $\mu: \mathcal{F} \to [0, +\infty]$ is a <u>measure</u>, i.e. $\mu(\phi) = 0$; $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for disjoint $A_i \in \mathcal{F}$.

Sets
- Measurable sets
- open, closed, compact
- F_σ, G_δ, Borel.
- $\limsup A_n$, $\liminf A_n$

[Question: What if measure is NOT defined in this way?]

There are many related conceptions:
- Outer measure ($\to$ Caratheodory measurability)
- Premeasure ($\to$ Elementary "measure", Jordan "measure", product measure)
- Metric outer measure ($\to$ e.g. Hausdorff measure) (G_δ, F_σ, ...)
- Borel measure
- Radon measure ($\to$ <u>outer regular</u>, <u>inner regular</u>, locally finite measure)
- Signed measure ($\to$ mutually singular, absolute continuous) $\to$ (Radon-Nikodym)
  $\to$ (Hahn decomposition, Jordan decomposition)
- complex-valued measure .....

Basic properties of measures
- monotonicity
- monotone convergence theorem. (increasing, decreasing)
- Fatou's lemma: $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \mu(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} \mu(A_n)$
- Dominated convergence theorem.

Important measures
- The Lebesgue measure m on $\mathbb{R}^d$
- The counting measure $\#$ on $\mathbb{N}$ or $\mathbb{Z}$
- The dirac measure δ at $x_0 \in \mathbb{R}^d$.

Abstract properties has their roots in concrete examples

- **Functions** We learned many different classes of functions
  - Characteristic functions χ_A
  - Step functions
  - Simple functions
  - Measurable functions
  - Integral functions, L^p functions $(1 \le p \le \infty)$
  - Smooth functions, continuous functions.
  - Compactly supported functions.
  - non-negative functions

- monotone functions
- BV functions
- AC functions.
- Lipschitz functions

Littlewood's three principle

Common sense: Use ~~[illegible]~~ simple functions to "approximate" complicated functions

$\rightsquigarrow$ different modes of convergence:
- pointwise convergence
- a.e. convergence
- absolute convergence
- convergence in measure
- convergence in L^p-norm

relation?

Integration $\int_A f \, d\mu$. $\leftarrow$ Three elements: A, f, μ.

- Non-negative theory
  - $\rightarrow$ Monotone convergence theorem
  - $\rightarrow$ Fatou's lemma
  - $\rightarrow$ Countable additivity (w.r.t. A or f or μ)
  - $\rightarrow$ Tonelli's theorem
- Absolute ~~[illegible]~~ integral theory $(f = f_+ - f_-)$
  - $\rightarrow$ Dominated convergence theorem
  - $\rightarrow$ Absolute continuity.
  - $\rightarrow$ Fubini's theorem.
- Lebesgue integral vs. Riemann integral
  - $\rightarrow \int_{[a,b]} f = \int_a^b f(x)\,dx$ if f is bounded and Riemann integrable $\Updownarrow$ $m(\{f \text{ is discontinuous at } x\}) = 0$
  - $\rightarrow \int_A |f(x)|^p dx = p \int_0^\infty t^{p-1} \cdot m(\{x : |f(x)| > t\})\,dt$

[Question. What is $\int_A g(|f(x)|)\,dx$ for nice g? e.g. monotone increasing.]

- Convolution: $\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}$
- multiplication: Hölder's inequality $\|fg\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}$

$\boxed{\text{Spaces of functions}}$

- Linear structure: Minkowski inequality $\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$.
- Metric structure: — ,

$\rightsquigarrow$ completeness: Any Cauchy sequence in $L^p(X)$ converges. $\boxed{L^\infty(X): \text{essentially bounded}}$

($\Rightarrow$ $L^p(X)$ is a Banach space, $1 \le p \le \infty$)

$\rightsquigarrow$ $L^p(\mathbb{R}^d)$ is separable for $1 \le p < \infty$.

↑ countable dense subset.

($\Leftarrow$: "Simple functions" is dense in $L^p(X)$, $\forall 0 < p < \infty$)

- Inner product space structure: $L^2(X)$. ($\Rightarrow$ Hilbert space)

"angle" "minimizer" "orthogonal decomposition":

- The dual spaces ($\leftarrow$ For linear functionals, "continuous" = "bounded")
  - $(L^2)^* = L^2$, $(H)^* = H$.
  - $(L^p)^* = L^q$, $1 \le p < \infty$ $\left(\frac{1}{p} + \frac{1}{q} = \infty\right)$
  - $(C(X))^*$ = "signed Borel measures" (X: compact)

"Riesz representation thms".

$\boxed{\text{The fundamental theorem of calculus}}$

- a.e. differentiable: · monotone functions ($\hookrightarrow \int_{[a,b]} f'\,dx \le f(b) - f(a)$)
  - · BV functions, AC functions
  - · Lipschitz functions. ($\hookrightarrow$ Lipschitz maps)
- The Lebesgue differentiation theorem: $f \in L^1_{loc}(\mathbb{R}^d) \Rightarrow$ for a.e. $x \in \mathbb{R}^d$
  - · $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - f(t)|\,dt = 0$.
  - · $\lim_{r \to 0} \frac{\mu_f(B(x,r))}{m(B(x,r))} = f(x)$

  $\Leftarrow$ Hardy-Littlewood

  $\Leftarrow$ Vitali covering.

  ("differentiation of measure"
  $$\lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = \frac{d\mu_r}{dm}(x)$$
  for locally finite outer regular measure μ.)

- The fundamental theorem of calculus

① $f \in L^1(\mathbb{R}) \Rightarrow \frac{d}{dx}\left(\int_{(-\infty, x)} f(t)\,dt\right) = f(x)$ a.e.

② $F: [a,b] \to \mathbb{R}$ is AC $\Rightarrow \int_{[a,b]} F'\,dt = F(b) - F(a)$.

Some Problem-Solving strategies

- One equality $\Leftrightarrow$ Two inequalities
  - $A=B \Leftrightarrow A \geq B$ and $A \leq B$.
  - $A=B$ as sets $\Leftrightarrow A \supset B$ and $A \subset B$.
- Convert "properties of functions" to "properties of sets"
  - $A \longleftrightarrow \chi_A$
  - $\limsup_{n\to\infty} f_n(x) > t \Longleftrightarrow x \in \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x : f_n(x) > t\}$ etc.
- Write one set as the union of countable sets.

  [Also, write uncountable union as countable union] [continuous limit as sequential limit]
  - $\{x : f(x) > 0\} = \bigcup_n \{x : f(x) > \frac{1}{n}\}$. $\lim_{h\to 0} \rightsquigarrow \lim_{\mathbb{Q}\ni h \to 0}$
  - $\{f(x) > t - g(x)\} = \bigcup_{r\in\mathbb{Q}} \{f(x) > r > t - g(x)\}$
- To control a quantity (e.g. an integral), use triangle inequality to split it into two parts which can be controlled using different methods.
- "take complement", or "$f \to -f$".
- Try simple objects first.
  - For measurable sets, try open/closed/compact first
  - For unbounded ~~sets~~/functions, try bounded sets/functions first
  - For measurable functions, try smooth/continuous/simple function first
  - For integrable functions, try non-negative/simple functions first
  - ...
- Last, but not least: give yourself an ε of room.
  - To prove $a=0$: $|a| < \varepsilon$, $\forall \varepsilon$
  - To prove $a \leq b$: $a < b + \varepsilon$, $\forall \varepsilon$.
  - sometimes $\varepsilon = \frac{\varepsilon}{n} + \cdots + \frac{\varepsilon}{n}$.
  - sometimes $\varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \cdots$

庄子《天下篇》
一尺之棰
日取其半
万世不竭

改编自
交大
来米加
版本

昨夜西风凋碧树，可数可加 σ 代数
众里寻他千百度，单调控制和 Fatou
衣带渐宽终不悔，可积可微几乎处处
为 ε 消得人憔悴，现代分析始上路