实分析讲义

$\S$ 3.1

1、Each Borel set is Lebesgue measurable

Pf：由 Borel 定义 $\mathcal{B} = \sigma(\{O \subset \mathbb{R} : O$ 为开集$\})$ 为包含开集的最小 σ 代数

易见 $\mathcal{L}$ 为 σ 代数，且 $\forall$ 开集 $O \in \mathcal{L}$

故 $\mathcal{B} \subseteq \mathcal{L}$

即 ~~$\forall$开集~~ $\forall$ Borel 集可测

3、$L : \mathbb{R}^d \to \mathbb{R}^d$ linear，Pf：$A \subset \mathbb{R}^d$，$A \in \mathcal{L} \Rightarrow L(A) \in \mathcal{L}$

~~设 $det(L) \neq 0$，即 L 可逆~~

~~Pf：法二：对 $\forall$ Box B，有 $m(L(B)) = |det(L)| m(B)$~~

~~半开半闭 即 $B_n = \coprod (a_i, b_i]$，故 $B_i \cap B_j = \phi$~~

~~$\forall$开集 $O = \bigcup_{n=1}^{\infty} B_n$ (B_n almost disjoint Box)~~ ~~由 L 可逆，$L(B_i) \cap L(B_j) = \phi$~~

~~$m(L(O)) = m(L(\bigcup B_n)) = m(\bigcup_n L(B_n))$~~

证 $\forall A \subset \mathbb{R}^d$，$m^*(L(A)) = \cancel{|det A|}\ |det L|\, m^*(A)$

Pf：若 $det(L) = 0$，即 $dim(L(\mathbb{R}^d)) \le d-1$ $\therefore m(L(\mathbb{R}^d)) = 0$

$\therefore \forall A \subset \mathbb{R}^d$ $m(L(A)) = |det(A)|\, m(A)$

若 $det(L) \neq 0$，即 L 可逆

对于 $\forall$ Box $B \subset \mathbb{R}^d$，由线代知识（$J(L(B)) = |det L| J(B)$

$m(L(B)) = |det L|\, m(B)$

对于 $\forall$开集 O，由于 L 可逆，故 ~~$L(B)$为~~ $L(O)$ 为开集

设 $O = \bigcup_n B_n$ $B_n = \coprod (a_i, b_i]$，$B_i \cap B_j = \phi$

则 $m(L(O)) = m(\bigcup_n L(B_n)) = \sum_n m(L(B_n)) = |det L| \sum_n m(B_n)$

$= |det L|\, m(O)$

对于 $\forall$集合 A

~~$m^*(A) = in$~~ $m^*(L(A)) = \inf\limits_{\text{开}O \supset L(A)} m^*(O) = \inf\limits_{L^{-1}(O) \supset A} m^*(O)$

$= \inf\limits_{\text{开}O \supset A} m^*(L(O)) = |det L| \inf\limits_{\text{开}O \supset A} m^*(O)$

$= |det L|\, m^*(A)$

Pf of 3

若 $\det L = 0$, $\forall A \in \mathcal{L}(\mathbb{R}^d)$ $\quad m^*(L(A)) = 0 \Rightarrow L(A) \in \mathcal{L}(\mathbb{R}^d)$

若 $\det L \neq 0$, $\forall \varepsilon > 0$

$\exists$开集 $O \supset A$, $m^*(O \setminus A) < \dfrac{\varepsilon}{|\det L|}$

由 L 可逆, $L(O)$ 为开集, $L(A) \subset L(O)$

且 $m^*(L(O) \setminus L(A)) = m^*(L(O \setminus A)) < \varepsilon$

$\therefore$ $L(A)$ 可逆

做法2: Lemma: 若连续函数 $f: \mathbb{R}^m \to \mathbb{R}^n$ 将零测集映为零测集, 则 f 将可测集映为可测集

~~$\forall A \in \mathcal{L}(\mathbb{R}^m)$ 设 A 有界~~

$\forall A \in \mathcal{L}(\mathbb{R}^m)$ 设 $A = F \cup Z$, $F \in F_\sigma$, $m(Z) = 0$

我们证 $f(F)$ 为 F_σ 设 $F = \bigcup_i \overset{闭}{F_i} = \bigcup_n \left(\bigcup_i F_i \cap \overline{B(0,n)}\right) = \bigcup_n \bigcup_i \left(F_i \cap \overline{B(0,n)}\right)$

Claim: 连续函数的性质: 将紧集映为紧集

(~~开映像定理: $\forall$开集~~ $O \subset$

(闭 $\to$ 闭 × 例 $F = \{(x,y): xy = 1\}$ 为闭集

$f: (x,y) \longmapsto (x,0)$ $\mathbb{R}^2 \to \mathbb{R}^2$

则 $f(F) = (-\infty, 0) \cup (0, +\infty)$ 开)

对紧集 A, 设 $f(A) \subset \bigcup_\alpha O_\alpha$

$\{O_\alpha\}$ 为 $f(A)$ 任一开覆盖

则 $A \subset f^{-1}(\bigcup O_\alpha) = \bigcup f^{-1}(O_\alpha)$ $\quad f$ 连续 $\Rightarrow f^{-1}(O_\alpha)$ 开

$\therefore$ $\exists$有限子覆盖 $A \subset \bigcup_{i=1}^n f^{-1}(O_i) = f^{-1}\left(\bigcup_{i=1}^n O_i\right)$

$\therefore f(A) \subset \bigcup_{i=1}^n O_i$ $\quad \therefore$ A 紧

由 Claim, $f(F_i \cap \overline{B(0,n)})$ 为紧集 $\therefore f(F) = \bigcup_n \bigcup_i f(F_i \cap \overline{B(0,n)})$ 为 F

$\therefore f(A) = f(F) \cup f(Z)$ $\quad f(F) \in F_\sigma$ 可测 $\quad f(Z)$ 可测

$\therefore f(A)$ 可测

~~keu. find f, f将正测集映为零测集~~

4. ~~挖去~~ $A \subset [0,1]$ which consists of all numbers which do not have the digit 6 appearing in their decimal expansion, what is $m(A)$

Pf: 仿造 Cantor 集，每次将集合分成 10 份，挖去第 7 个开区间

例：第一次挖去 $(\frac{6}{10}, \frac{7}{10})$ 此时集合为 A_1

在第二次时，每个小区间分成 10 份，挖去第 7 个开区间

即挖去 $(\frac{6}{100}, \frac{7}{100}) + \frac{i}{10}$ $i = 0, 1, \cdots, 5, 7, 8, 9$ 为 A_2

$$A = \bigcap_{n=1}^{\infty} A_n \quad 而 \quad m(A_n) = \frac{9}{10} m(A_{n-1}) = \left(\frac{9}{10}\right)^n$$

$$\therefore m(A) \leq A_N = \left(\frac{9}{10}\right)^N \Rightarrow m(A) = 0$$

5. $A_1, A_2 \in \mathcal{L}(\mathbb{R}^d)$, $m(A_1) > 0$, $m(A_2) > 0$,

Prove: $A_1 + A_2 = \{x + y \mid x \in A_1, y \in A_2\}$ contains a box

~~3.2~~ ~~函数~~ f_n 可测，~~Pf: f 的 $C(f)$ 可测~~ 连续点集 Pf: f_n 收敛点集可测

~~$C(f) = \{x:$~~

6. $\{x : f_n(x) 收敛\} = \{x : \forall \varepsilon > 0, \exists N > 0, \forall n, m > N, |f_n(x) - f_m(x)| < \varepsilon\}$

$$= \bigcap_{n=1}^{\infty} \{x : \exists N > 0, \forall k, l > N, |f_k - f_l| < \tfrac{1}{n}\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \{x : \forall k, l > N, |f_k - f_l| < \tfrac{1}{n}\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \bigcap_{l=N}^{\infty} \{x : -\tfrac{1}{n} < f_k - f_l < \tfrac{1}{n}\} \in \mathcal{L}$$

或：$\limsup f_n$、$\liminf f_n$ 可测

$\{收敛点\} = \{x : \overline{\lim} f_n = \underline{\lim} f_n\}$ 可测

$$\limsup f_n = \lim_{N \to \infty} (\sup_{n > N} f_n)$$

~~而 $\tilde{G}_\infty = \bigcup_{i=1}^{\infty} G_i \supset \bigcup_{n=1}^{\infty} A_n$~~

设 G 为 $\bigcup A_n$ 等测包

$A_i \subseteq G \cap \tilde{G}_i \subseteq \tilde{G}_i$

$\therefore m(G \cap \tilde{G}_i) = m^*(A_i)$

而 $\bigcup (G \cap \tilde{G}_i) = G \cap \tilde{G}_\infty$

$\bigcup_n A_n \subset G \cap \tilde{G}_\infty \subset G \ldots$

7. $A_1 \subset A_2 \subset \cdots \subset \mathbb{R}^d$，证 $\lim_{n \to \infty} m^*(A_n) = m^*(\bigcup_{n=1}^{\infty} A_n)$

证：等测包 $G_i \supset A_i$，$m^*(A_i) = m(G_i)$ 令 $\tilde{G}_i = \bigcup_{j \leq i} G_j$ 则 $A_i \subseteq \tilde{G}_i$

~~$\forall i < j$ G_i $m(\tilde{G}_1) = m(G_1)$ 若 $m(\tilde{G}_{k-1}) = m(G_{k-1})$ $\tilde{G}_k = \tilde{G}_{k-1} \cup G_k$ $\tilde{G}_k \setminus G_k = \tilde{G}_{k-1} \setminus G_k$~~

$\forall i < j$, $m(G_i \setminus G_j) = m(G_i) - m(G_i \cap G_j) = 0$，注：$A_i \subset G_i \cap G_j \subset G_i$ $\tilde{G}_i \setminus G_i = \tilde{G}_{i-1} \setminus G_i$ 零测

2 $(A \in \mathcal{L})$

① $f: A \subset \mathbb{R}^d \to \mathbb{R}$ is measurable

$\Leftrightarrow$ ② $\forall$ open $U \subset \mathbb{R}$ $f^{-1}(U) \in \mathcal{L}$

$\Leftrightarrow$ ③ $\forall$ closed $F \subset \mathbb{R}$ $f^{-1}(F) \in \mathcal{L}$

证： ② ⇒ ①，~~③~~ 显然（取 $U = (a, +\infty)$）

① ⇒ ②，$\forall$ 开集 $U = \bigcup_i (a_i, b_i)$

$$f^{-1}(U) = \bigcup_i f^{-1}(a_i, b_i) = \bigcup_i \left(f^{-1}(a_i, +\infty) \cap f^{-1}(-\infty, b_i)\right) \in \mathcal{L}$$

② ⇒ ③，$\forall F \subset \mathbb{R}$, F^c 开，$f^{-1}(F) = \left(f^{-1}(F^c)\right)^c \in \mathcal{L}$

③ ⇒ ②，$\forall$ 开 $U \in \mathbb{R}$, U^c 闭，$f^{-1}(U) = \left(f^{-1}(U^c)\right)^c \in \mathcal{L}$

2. $f, g \in \mathcal{L}(A)$. Pf: $\{x : f > g\}, \{x : f \geq g\}, \{x : f = g\} \in \mathcal{L}$

Pf: $\{x : f > g\} = \bigcup_{r \in \mathbb{Q}} \{x : f(x) > r > g(x)\} = \bigcup_{r \in \mathbb{Q}} \left(\{x : f > r\} \cap \{x : g < r\}\right) \in \mathcal{L}$

$\{x : f \geq g\} = \{x : f < g\}^c \in \mathcal{L}$

$\{x : f = g\} = \{x : f \geq g\} \cap \{x : f \leq g\} \in \mathcal{L}$

3. $f: A \to \mathbb{C}$ measurable ($\mathrm{Re} f$. $\mathrm{Im} f$ $\in \mathcal{L}(A)$), ① $\Leftrightarrow$ $\forall$ 开 $U \subset \mathbb{C} = \mathbb{R}^2$, $f^{-1}(U) \in \mathcal{L}$

Pf: "⇒" ~~由~~ $U = \bigcup_{i=1}^{\infty} B_i$ B_i box, almost disjoint $B_i = [a_i, b_i] \times [c_i, d_i]$

$$f^{-1}(U) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) = \bigcup_{i=1}^{\infty} \left\{ (\mathrm{Re} f)^{-1}([a_i, b_i]) \cap (\mathrm{Im} f)^{-1}([c_i, d_i]) \right\} \in \mathcal{L}$$

"⇐" 令 $\forall a$, $U = \{z : \mathrm{Re}\, z > a\}$ 开

则 $(\mathrm{Re} f)^{-1}((a, +\infty)) = f^{-1}(U) \in \mathcal{L}$ ⇒ $\mathrm{Re} f \in \mathcal{L}$

同理 $\mathrm{Im} f \in \mathcal{L}$ ⇒ $f \in \mathcal{L}$

$m(A)$

—key: find f, f将正测集映为零测集

4. $f: A \to \mathbb{R}$ measurable, $C \subset \mathbb{R}$ measurable, $f^{-1}(C) \in \mathcal{L}$?

Pf: 反例: 设 $g(x) = \frac{x+\Phi(x)}{2}$ 则 g 严格增, $\therefore$ g可测, 可逆, g^{-1}可测

设 C 为Cantor 集,则 $m(g(C)) > 0$ 令 $f = g^{-1}$

$\therefore \exists$ 不可测 $W \subset g(C)$ (*)

则 $f(W) \subset f \circ g(C) = C \Rightarrow f(W)$零测, 可测

但 $f^{-1}(f(W))$ 不可测

(*) 正测集具有不可测子集:

设 $A \in \mathcal{L}$, $m(A) > 0$, 令 $W = A/\mathbb{Q}$, 不妨设 $A \subset [0,1]$

即在 A上定义$\sim$关系

$x \sim y \Leftrightarrow x - y \in \mathbb{Q}$, ~~则 $A = U$~~

令 ~~W为~~ 在每个等价类中取一个元素构成W,

若 W可测, 令 $W_r = W + r$, $r \in \mathbb{Q} \cap \cancel{[0,1]}\ [-1,1]$

则 $A \subset \bigcup_{r \in \mathbb{Q} \cap [-1,1]} W_r \subset [-1, 2]$

即 $0 < m(A) \le \sum_r m(W_r) \le 3$, 这不可能

5. Borel $C \subset \mathbb{R}$, $f^{-1}(C) \in \mathcal{L}$? f可测

Pf: 令 $\Sigma := \{A \subset \mathbb{R} : f^{-1}(A) \in \mathcal{L}\}$

则 ① $f^{-1}(\mathbb{R}) \in \mathcal{L} \Rightarrow \mathbb{R} \in \Sigma$

② $A \in \Sigma$, 则 $f^{-1}(A^c) = f^{-1}(A)^c \in \mathcal{L}$

$\Rightarrow A^c \in \Sigma$

③ $A_1, A_2, \cdots \in \Sigma$ 即 $f^{-1}(A_n) \in \mathcal{L}$

则 $f^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_i f^{-1}(A_i) \in \mathcal{L}$

$\therefore \bigcup_i A_i \in \Sigma$

$\therefore \Sigma$ 为 σ 代数, 而 $\forall$开集 $O \in \Sigma \Rightarrow \mathcal{B} \subset \Sigma$

$\therefore \forall$ Borel C, $f^{-1}(C) \in \mathcal{L}$

4.1

1. Borel - Cantelli Lemma

$\sum_{k=1}^{\infty} m(A_k) < \infty \Rightarrow \limsup_{k\to\infty} A_k = \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} A_k$ 零测

证：$\forall N$,

$$m(\limsup_{k\to\infty} A_k) \leq m(\bigcup_{k=N}^{\infty} A_k) \leq \sum_{k=N}^{\infty} m(A_k)$$

$$N\to\infty \Rightarrow m(\limsup_{k\to\infty} A_k) = 0$$

2. $f_n \to f$，$g_n \to g$ in measure

(1) $f_n+g_n \to f+g$ in measure

(2) $m(A)<\infty \Rightarrow f_n g_n \to fg$ in measure

(3) can't drop $m(A)<\infty$

证：(1) $\{|f_n+g_n-f-g|>\varepsilon\} \subset \{|f_n-f|+|g_n-g|>\varepsilon\}$

$\subset \{|f_n-f|>\frac{\varepsilon}{2}\} \cup \{|g_n-g|>\frac{\varepsilon}{2}\}$

$\therefore m(|f_n+g_n-f-g|>\varepsilon) \leq m(|f_n-f|>\frac{\varepsilon}{2}) + m(|g_n-g|>\frac{\varepsilon}{2}) \to 0$

$\therefore f_n+g_n \to f+g$ in measure

(2) $\{|f_ng_n-fg|>\varepsilon\} \subseteq \{|f_n||g_n-g|>\frac{\varepsilon}{2}\} \cup \{|g||f_n-f|>\frac{\varepsilon}{2}\}$

$\{|g||f_n-f|>\frac{\varepsilon}{2}\} \subseteq \{|g|>M\} \cup \{|f_n-f|>\frac{\varepsilon}{2M}\}$

$\therefore m(|g||f_n-f|>\frac{\varepsilon}{2}) \leq m(|g|>M) + m(|f_n-f|>\frac{\varepsilon}{2M})$

令 $n\to\infty \Rightarrow \lim_{n\to\infty} m(|g||f_n-f|>\frac{\varepsilon}{2}) \leq m(|g|>M)$

由 $m(A)<\infty \Rightarrow \lim_{M\to\infty} m(|g|>M)=0$ （$\{|g|>1\} \supset \{|g|>2\} \supset \cdots\cdots$

$\therefore$ 令 $M\to\infty \Rightarrow \lim_{n\to\infty} m(|g||f_n-f|>\frac{\varepsilon}{2})=0$ $\quad \bigcap_N \{|g|>N\} = \{g=\infty\}$ 零测）

$\{|f_n||g_n-g|>\frac{\varepsilon}{2}\} \subseteq \{|f_n|>M\} \cup \{|g_n-g|>\frac{\varepsilon}{2M}\} \subseteq \{|f_n-f|>\frac{M}{2}\} \cup \{|f|>\frac{M}{2}\} \cup \{|g_n-g|>\frac{\varepsilon}{2M}\}$

$\therefore m(|f_n||g_n-g|>\frac{\varepsilon}{2}) \leq m(|f_n-f|>\frac{M}{2}) + m(|g_n-g|>\frac{\varepsilon}{2M}) + m(|f|>\frac{M}{2})$

先令 $n\to\infty$，再令 $M\to\infty$，得 $m(|f_n||g_n-g|>\frac{\varepsilon}{2}) \to 0$ $\therefore m(|f_ng_n-fg|>\varepsilon) \to 0$

(3) 令 ~~n~~ $\varepsilon=1$ $f_n \to x$, $g_n \to 0$

则 $m(|f_n g_n|>1) = m(n,+\infty) = \infty$

~~令n~~ $\therefore \lim_{n\to\infty} m(|f_n g_n|>1) = \infty$, . . .

3. $f_n - f$ almost uniformly

Pf: $f_n \to f$ a.e. $f_n - f$ in measure

Pf: ① 取 A_n, $m(A_n) < \frac{1}{n}$, $f_n \to f$ uniformly on $A \setminus A_n$

$\therefore f_n \to f$ ~~on a.e.~~ pointwise on $A\setminus A_n$

$\therefore f_n \to f$ pointwise on ~~$\emptyset$~~ $\bigcup_n (A\setminus A_n) = A \setminus \bigcap_n A_n$

而 $m(\bigcap A_n) = 0$

$\therefore f_n \to f$ a.e. on A

~~② $\forall \varepsilon>0$ $m(|f_n - f|>\varepsilon) \to 0$~~

~~$m(\{|f_n - f|>\varepsilon\}\setminus A_m) \to 0$~~

~~即 $m\{|f_n - f|>\varepsilon\}\setminus A_m\} < \frac{1}{N}$~~

② 反证, $\exists \varepsilon_0 > 0$. $\{f_{n_j}\} \subset \{f_n\}$ $\delta > 0$

$m(|f_{n_j} - f| > \varepsilon_0) \geq \delta$

取 A_δ, $m(A_\delta) < \delta$, $A\setminus A_\delta$ 上 $f_n \rightrightarrows f$ ~~uni~~

$\therefore$ n 足够大 $\{|f_n - f| > \varepsilon_0\} \subseteq A_\delta$

$\therefore \lim_{n\to\infty} m(|f_n - f| > \varepsilon_0) \leq m(A_\delta) < \delta$, 矛盾

4. f, simple

Pf: $\forall \varepsilon > 0$, $\exists$ close $F \subset A$, $m(A\setminus F) < \varepsilon$, $g \in C(\mathbb{R})$, $f = g$ on F

设 $f = \sum_{i=1}^{n} c_i \chi_{A_i}$ $A_i \cap A_j = \emptyset$

~~其~~ $\exists$ 闭 $F_i \subseteq A_i$, $m(A_i \setminus F_i) < \frac{\varepsilon}{n}$ 令 $F = \bigcup_i$ ~~A_i~~ F_i 则 $m(A\setminus F) < \varepsilon$

将 $f|_F$ 扩张 . . .

~~扩张~~ 需证 $f|_F$ 连续 ~~在~~ f 在 $F\cap\overline{B(0,n)}$ 上连续 $\Rightarrow f$ 在 F 上连续

($F_i \cap \overline{B(0,n)}$ 紧, 两两不交, $\Rightarrow d(F_i \cap \overline{B(0,n)}, F_j \cap \overline{B(0,n)}) >$

4.2.

1. g 可测，∃ semisimple $f_n \rightrightarrows g$

证：令 $C_k = \frac{k}{2^n}$，$A_k = \{x : \frac{k}{2^n} \le g < \frac{k+1}{2^n}\}$

令 $f_n = \sum_{k=-\infty}^{+\infty} C_k A_k$

则 $\sup|g - f_n| \le \frac{1}{2^n}$

$\therefore f_n \rightrightarrows g$

2、f 可测，∃ step $f_n \to f$ a.e.

证：只需对 $f = \chi_A$ 作证明，$A \in \mathcal{L}$

而 $m(A) = \inf_{E \text{ elem}} m(E)$　　$A \subseteq E_n$　　$m(E_n) - \frac{1}{2^n} < m(A)$，即 $m(E_n \setminus A) < \frac{1}{2^n}$

2题补充：

由证明过程，$\forall f$ simple，∃ f_n step s.t. $m(|f - f_n| > 0) < \frac{1}{2^n}$ ~~as lecture~~

设 $\forall f$，measurable，f_n simple ~~as lecture~~

$f_n \to f$ a.e. f_{nm} step

$m(|f_n - f_{nm}| > 0) < \frac{1}{2^m}$

On each $B(0,R)$ 令 $g_n = f_{nn}$

$m(|f - f_{nn}| > \varepsilon) \le m(|f - f_n| > \frac{\varepsilon}{2}) + \frac{1}{2^n} \to 0$

$\therefore$ ∃子列 $\{g_{n_j}^{(R)}\}$，$g_{n_j}^{(R)} \to f$ a.e. on $B(0,R)$

故取 $\{g_n\} \supseteq \{g_n^{(1)}\} \supseteq \{g_n^{(2)}\} \supseteq \cdots$ 则 $g_n^{(n)} \to f$ a.e.

令 $f_n = \chi_{E_n}$ 为 step

$\forall 0 < \varepsilon < 1$，$\{|f_n - f| > \varepsilon\} = E_n \setminus A$　$\because \sum_{n=1}^{\infty} m(|f_n - f| > \varepsilon) = \sum_{n=1}^{\infty} m(E_n \setminus A) = 1 < \infty$

$\therefore m(\limsup \{|f_n - f| > \varepsilon\}) = 0$

$\therefore f_n \to f$ ae

3. Ury + Tiez　　T_4 space

4. $\|f\|_{\mathcal{L}} = \inf\{r : m(\{x : |f(x)| > r\}) \le r\}$

① $m\{x : |f(x)| > \|f\|_{\mathcal{L}}\} \le \|f\|_{\mathcal{L}}$

② $\|f+g\|_{\mathcal{L}} \le \|f\|_{\mathcal{L}} + \|g\|_{\mathcal{L}}$

③ $f_n \to f$ in measure $\Leftrightarrow \|f_n - f\|_{\mathcal{L}} \to 0$

④ for any $A \in \mathcal{L}$, $c > 0$　$\|c\chi_A\|_{\mathcal{L}} = \inf\{c, m(A)\}$

Pf: ① $\forall$ $\exists\, r_n$ $\quad r_n-\frac{1}{n}<\|f\|_{\mathcal{L}}\le r_n$ $\quad \|f\|_{\mathcal{L}}=\infty$ 时显然，若 $\|f\|_{\mathcal{L}}<\infty$

不妨设 $r_n>r_{n+1}$

$$m(|f(x)|>r_n)\le r_n<\|f\|_{\mathcal{L}}+\frac{1}{n}$$

~~$\{|f(x)|$~~ 令 $n\to\infty$

有 $m(|f(x)|>\|f\|_{\mathcal{L}})\le\|f\|_{\mathcal{L}}$

②

~~$m(|f+g|>r)$~~

$$\|f\|_{\mathcal{L}}+\|g\|_{\mathcal{L}}\ge m\{|f|>\|f\|_{\mathcal{L}}\}+m(|g|>\|g\|_{\mathcal{L}})$$

$$\ge m\{|f|+|g|>\|f\|_{\mathcal{L}}+\|g\|_{\mathcal{L}}\}$$

$$\ge m\{|f+g|>\|f\|_{\mathcal{L}}+\|g\|_{\mathcal{L}}\}$$

由 $\|f+g\|_{\mathcal{L}}$ 定义 $\Rightarrow$ $\|f+g\|_{\mathcal{L}}\le\|f\|_{\mathcal{L}}+\|g\|_{\mathcal{L}}$

③ "$\Rightarrow$" $\forall\varepsilon>0$

$$m(|f_n-f|>\varepsilon)\to 0$$

$\exists N$, $n>N$时~~n足够大时~~ $\quad m(|f_n-f|>\varepsilon)<\varepsilon$

$\therefore \|f_n-f\|_{\mathcal{L}}\le\varepsilon$

$\therefore \|f_n-f\|_{\mathcal{L}}\to 0$

"$\Leftarrow$" ~~$\forall\varepsilon>0$~~

反证，$\exists\{f_{n_j}\}\subset\{f_n\}$，$\varepsilon_0,\delta>0$，不妨设 $\delta<\varepsilon_0$

$$m(|f_{n_j}-f|>\varepsilon_0)>\delta$$

但 $\|f_{n_j}-f\|_{\mathcal{L}}\to 0$

即 j 足够大时 $\quad \|f_{n_j}-f\|_{\mathcal{L}}<\delta$ $\quad \therefore \exists\delta'<\delta$ 使

~~$m(|f$~~

$m(|f_{n_j}-f|>\varepsilon_0)<m(|f_{n_j}-f|>\delta')<\delta'<\delta$ 矛盾

~~④~~ 若 $m(A)$

~~$\{r:\ m(\{c\chi_A>r\})<r\}\neq\emptyset$~~

~~易见，$0<r<c$ 时 $\quad m(\{c\chi_A>r\})=m(A)$~~

~~$\therefore m(A)\le c$ 时 $\quad m(\{c\chi_A>m(A)\})=m(A)$ $\quad\therefore \|c\chi_A\|_{\mathcal{L}}\le m(A)$~~

~~$\forall 0<r<m(A)$ $\quad m(\{c\chi_A>r\})=m(A)>r$，$\therefore \|c\chi_A\|_{\mathcal{L}}\ge r\Rightarrow\|c\chi_A\|_{\mathcal{L}}=m(A)$~~

~~若 $m(A)>c$ $\quad\forall m(A)$~~

④ ~~$\forall\ 0 < r < m(A$ c~~ $f = c\chi_A$

~~$m(\{c\chi_A > r\}) = m(A)$~~

$\forall\ r \geq c \quad m(\{|c\chi_A| > r\}) = m(\phi) = 0 < r \Rightarrow \|f\|_{\mathcal{L}} \leq c$

而 $m(\{c\chi_A > m(A)\}) = \begin{cases} m(\phi) = 0 \leq m(A) & \text{if } c \leq m(A) \\ m(A) & \text{if } c > m(A) \end{cases}$

$\therefore \|f\|_{\mathcal{L}} \leq m(A)$

$\therefore \|f\|_{\mathcal{L}} \leq \inf\{c, m(A)\}$

$\forall\ r < \inf\{c, m(A)\}$

有 $m(c\chi_A > r) = m(A) > r \quad \therefore r \notin \{r : m(c\chi_A > r) < r\}$

$\therefore \|f\|_{\mathcal{L}} \geq r$

由 r 取值 $\|f\|_{\mathcal{L}} \geq \inf\{c, m(A)\}$

$\therefore \|f\|_{\mathcal{L}} = \inf\{c, m(A)\}$

$f_{n_i} \rightarrow f_n \quad f_n \rightarrow f$

$f - \alpha f_n$

$$\begin{array}{cccccc} f_{11} & f_{12} & \cdots & \rightarrow & f_1 \\ f_{21} & f_{22} & \cdots & \rightarrow & f_2 \\ \vdots & \vdots & & & \vdots \\ & & & & f \end{array}$$

$n \geq N_1 \qquad n \geq N_2$

$|f - f_{nn}| < |f - f_n| + |f_n - f_{nn}|$

$< \varepsilon \qquad < \frac{\varepsilon}{2} \qquad < \frac{\varepsilon}{2}$