Thm: g 是 $E\subset\mathbb{R}^n$ 上可测，$\exists M>0$，使 $\forall$ simple 可积 φ，有 $\frac{1}{p}+\frac{1}{q}=1$

$1\le p<\infty$ $\quad \left|\int_E g(x)\varphi(x)\,dx\right|\le M\|\varphi\|_p$

则 $g\in L^q(E)$，$\|g\|_q\le M$

Pf: (i) $p>1$ 时，

$\exists$ 紧支非负可测简单升列 $\{\varphi_k\}$

$$\varphi_k\to|g|^q \quad a.e$$

令 $\quad \psi_k=[\varphi_k]^{\frac{1}{p}}\operatorname{sign} g$

则 $\quad \|\psi_k\|_p=\left(\int_E\varphi_k(x)dx\right)^{\frac{1}{p}}=\cancel{\|\varphi_k\|_p}$

而 $\quad 0\le\varphi_k\cancel{(x)}=\cancel{\varphi_k(x)}(\varphi_k)^{\frac{1}{p}}(\varphi_k)^{\frac{1}{q}}\le(\varphi_k)^{\frac{1}{p}}g=\psi_k g$

由假设：$\int_E\varphi_k\,dx\le\int_E\psi_k g\,dx\le M\|\psi_k\|_p$

$$\therefore\left(\int_E\varphi_k(x)dx\right)^{\frac{1}{q}}\le M$$

令 $k\to\infty$，由 MCT

$$\|g\|_q\le M$$

(ii) $p=1$，反证

若 $\|g\|_{L^\infty}>M$

~~$\therefore\exists\{A_k\}\ \infty>m(A_k)>0$~~

$\exists A,k\quad 0<m(A)<\infty$，$|g|>M+\frac{1}{k}$ on A

~~则~~ $\quad\varphi=\chi_{A}\operatorname{sign} g$

则 $\int_E\cancel{\chi_A}\varphi g\,dx=\int_A|g|dx>(M+\frac{1}{k})m(A)$

$=(M+\frac{1}{k})\|\varphi\|_p$ 矛盾！

注：$L^q(\mathbb{R}^n)$ 是 $L^p(\mathbb{R}^n)$ 的对偶空间 $(1\le p<\infty)$

即 $\forall$ 有界线性泛函 $\mathcal{L}$ ~~对~~ : $L^p(\mathbb{R}^n)\to\mathbb{R}$

$\exists g\in L^q(\mathbb{R}^n)$ s.t. $\forall f\in L^p(\mathbb{R}^n)$，$\mathcal{L}(f)=\int fg\,dx$

广义 Minkowski ineq

$f(x,y)$ 在 $\mathbb{R}^n \times \mathbb{R}^n$ 上可测. ~~(X,μ). (Y,ν)~~

对 a.e. $y \in \cancel{\mathbb{R}^n}\ Y$ $f(x,y) \in \cancel{L^p(\mathbb{R}^n X, \mu)}\ L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$

则 $\left[\int_X \left|\int f(x,y)\,dy\right|^p dx\right]^{\frac{1}{p}} \le \int \left(\int |f(x,y)|^p dx\right)^{\frac{1}{p}} dy$

证: 不妨设 $p>1$, 右边 $<\infty$

令 $g(x) = \int_{\mathbb{R}^n} |f(x,y)|\,dy$

$\forall$ 简单可积 φ, 有:

$$\left|\int_{\mathbb{R}^n} g\varphi\,dx\right| \le \int_{\mathbb{R}^n} |g\varphi|\,dx$$

$$= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |f(x,y)|\,dy\right]|\varphi(x)|\,dx$$

$$= \int\int |f(x,y)\varphi(x)|\,dx\,dy$$

$$\le \int\left[\int |f(x,y)|^p dx\right]^{\frac{1}{p}}\left[\int |\varphi(x)|^q dx\right]^{\frac{1}{q}} dy$$

$$= M\|\varphi\|_q$$

$\therefore$ $\|F\|_p \le M$

即 $\left\{\int \left|\int f(x,y)\,dy\right|^p dx\right\}^{\frac{1}{p}} \le \int\left[\int |f(x,y)|^p dx\right]^{\frac{1}{p}} dy$

卷积:

Young ineq

$f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$

则 $\|f*g\|_p \le \|f\|_1 \|g\|_p$

证: $|(f*g)(x)| \le \int_{\mathbb{R}^n} |f(x-y)||g(y)|\,dy$

$$= \int_{\mathbb{R}^n} |f(x-y)|^{\frac{1}{p}}|g(y)|\,|f(x-y)|^{\frac{1}{q}}\,dy$$

$$\le \left[\int |f(x-y)||g(y)|^p dy\right]^{\frac{1}{p}}\left[\int |f(x-y)|\,dy\right]^{\frac{1}{q}}$$

$$\therefore \left(\int |f*g(x)|^p dx\right)^{\frac{1}{p}} \le \|f\|_1^{\frac{1}{q}} \left[\int\int |f(x-y)||g(y)|^p \, dy\, dx\right]^{\frac{1}{p}}$$
$$= \|f\|_1^{\frac{1}{q}} \left[\int |g(y)|^p \left[\int |f(x-y)| dx\right] dy\right]^{\frac{1}{p}}$$
$$= \|f\|_1 \|g\|_p$$

Def: K 在 $\mathbb{R}^n$ 上, $\varepsilon > 0$

$$K_\varepsilon(x) = \varepsilon^{-n} K\left(\frac{x}{\varepsilon}\right)$$

Thm: $K \in L^1(\mathbb{R}^n)$ $\|K\|_1 = 1$, $f \in L^p(\mathbb{R}^n)$

则
$$\lim_{\varepsilon \to 0} \|K_\varepsilon * f - f\|_p = 0$$

$$(K_\varepsilon * f)(x) - f(x) = \int \{f(x-y) - f(x)\} K_\varepsilon(y) dy = \int \{f(x-\varepsilon y) - f(x)\} K(y) dy$$

$$\left\{\int |(K_\varepsilon * f)(x) - f(x)|^p dx\right\}^{\frac{1}{p}}$$

$$= \left\{\int \left|\int [f(x-\varepsilon y) - f(x)] K(y) dy\right|^p dx\right\}^{\frac{1}{p}}$$

$$\le \int \left\{\int |f(x-\varepsilon y) - f(x)|^p dx\right\}^{\frac{1}{p}} |K(y)| dy$$

令 $F_\varepsilon(y) = \left\{\int |f(x-\varepsilon y) - f(x)|^p dx\right\}^{\frac{1}{p}} \le 2\|f\|_p$

则 $0 \le F_\varepsilon(y)|K(y)| \le 2\|f\|_p |K(y)|$

$\varepsilon \to 0$ 时 $F_\varepsilon(y) \to 0$

由 DCT,

$$\lim_{\varepsilon \to 0} \|K_\varepsilon * f - f\|_p = 0$$

~~$K_\varepsilon =$~~ $K(x)$ ~~为~~ $\eta(x) = c\exp\left(-\frac{1}{1-|x|^2}\right)\chi_{\{|x| \le 1\}}$ 时

$\eta \in C_c^\infty(\mathbb{R}^n)$, "磨光算子" (mollifier)

且: $f \in L^1_{loc}(\mathbb{R}^n) \Rightarrow f^\varepsilon = f * \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$

Pf: $\forall x \in \mathbb{R}^n,\ i \in \{1,\dots,n\},\ |h|<1$

$$\frac{f^\varepsilon(x+he_i) - f^\varepsilon(x)}{h} = \frac{1}{\varepsilon^n}\int \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] f(y)\,dy$$

$$(*) = \frac{1}{\varepsilon^n}\int_V \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] f(y)\,dy$$

V 有界开, $V \supset \left\{y : \left|\frac{x+e_i-y}{\varepsilon}\right| < 1\right\} \cup \left\{y : \left|\frac{x-y}{\varepsilon}\right| < 1\right\}$

且 $\frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] \rightrightarrows \frac{1}{\varepsilon}\eta_{x_i}\left(\frac{x-y}{\varepsilon}\right)$ 一致

$\therefore\ (*) \xrightarrow{h\to 0} \int_V \eta_{\varepsilon, x_i}(x-y) f(y)\,dy = \int \eta_{\varepsilon,x_i}(x-y) f(y)\,dy$

类似地, $\forall\ \alpha$ 多重指标

$$D^\alpha f^\varepsilon(x) = (D^\alpha \eta_\varepsilon * f)(x)$$

$\therefore$ 各阶导数存在连续 $\Rightarrow f^\varepsilon \in C^\infty(\mathbb{R}^n)$

最后: $C_c^\infty(\mathbb{R}^n)$ 在 $L^p(\mathbb{R}^n)$ 中稠密

令 $f_N = f\chi_{B(0,N)}$, $f \in L^p(\mathbb{R}^n)$

则 $f_N^\varepsilon \in C_c^\infty(\mathbb{R}^n)$

且 $\|f - f_N^\varepsilon\|_p \le \|f_N^\varepsilon - f_N\|_p + \|f_N - f\|_p \to 0$

$\square$

Much Much More General

~~E∈~~ $C_c^\infty(\mathbb{R}^n)$ 在 $W^{k,p}$

~~期末~~

midterm:

5. "Danial Number" x contains 5410.

Pf: a.e. $(0,1)$ is Danial

Pf: ~~证~~ $m(\{\exists\ 5410\}) = 1$

事实上 $m(\{4k+1 \sim 4k+4 \text{ 位为 } 5410\}) = 1$ 记 A
$\exists k = 0, 1, \dots$

1~4 位上无 5410 . $m = \frac{9999}{10000}$

1~4 , 4~8 位上无 5410 : $m = \left(\frac{9999}{10000}\right)^2$

$\therefore\ m\{4k+1 \sim 4k+4 \text{ 无 } 5410, 0 \le k \le n\}) = \left(\frac{9999}{10000}\right)^{n+1}$

$\therefore\ m\{A^c\} = 0$

$\therefore\ m(\exists 5410) \geq m(A) = 1$

6. ~~$f_1 = \chi_f$~~ $A_1 = \{f > 1\}$ $f_1 = \chi_{A_1}$

$A_{n+1} = \{f > f_n + \frac{1}{n+1}\}$ $f_{n+1} = f_n + \frac{1}{n+1}\chi_{A_{n+1}}$

即 $f_n = \sum_{k=1}^{n} \frac{1}{n}\chi_{A_n}$ $f_n \nearrow$ $f_n \le f$ $\therefore f_n$ 收敛

~~$\therefore \|f - f_n\|$~~

~~$f - f_n \le \frac{1}{n}\chi_{A_n^c} + f\chi_{A_{n+1}}$~~ $f\chi_{A_n}$

~~$\forall x$, 令 $\{k_i : A_{k_i} \ni x\}$,~~ $I_x = \{k : x \in A_k\}$

~~证 $f(x) = \sum_{I_x} \frac{1}{k_i}$ 显然 $f(x) \geq \sum_{I_x} \frac{1}{k_i}$~~

~~可证 $\sum_{I_x} \frac{1}{k} \geq f(x)$ 即 $\forall n$, $\frac{1}{n} + \sum_{I_x} \frac{1}{k} \geq f(x)$~~

~~若不然，即 $\exists n$~~

~~$f(x) > \sum_{I_x} \frac{1}{k} + \frac{1}{n}$~~

~~$\forall x$，$\exists N_x$，s.t. $x \in A_1, \dots, A_{N_x}$ 但 $x \notin A_{N_x+1}$ 否则 $f(x) = \infty$~~

~~$\therefore \forall n > N_x$，$f(x) - f_n(x)$~~

固定 x，~~我们证 $\forall k$，$\exists N(k,x) \in \mathbb{N}_x$，s.t. $f(x) - f_n(x) < \frac{1}{k}$，~~

则 $\forall k$，$\exists n_k > k$ s.t. $x \notin A_{n_k}$

否则，$x \in \bigcup_{n>k} A_n$ 从而 $f(x) = \sum_{n>k} \frac{1}{n} = \infty$，矛盾

$\therefore x \notin A_{n_k} \Rightarrow f(x) - f_{n_k}(x) < \frac{1}{n_k}$

$\therefore f_{n_k}(x) \to f(x)$

$\therefore f_n(x) \to f(x)$

$\square$

7. $f_n \to f$ a.e. $f \in L^p(A)$

$$\int_A |f_n - f|^p dx \to 0 \Leftrightarrow \int_A |f_n|^p dx \to \int_A |f|^p dx$$

证："$\Rightarrow$" $\big| \|f_n\|_p - \|f\|_p \big| \le \|f_n - f\|_p$

"$\Leftarrow$" ~~$|f_n - f|^p \le 2|f| + 2|f_n|$ $2|f|^p$~~

$$|f_n(x) - f(x)|^p \le \max\{2^p |f(x)|^p, 2^p |f_n(x)|^p\} \le 2^p (|f(x)|^p + |f_n(x)|^p)$$

由DCT 可得

9. Vitali 收敛定理

lemma：$\{f_n\}$ 连续 (X, d)，$f(x) = \lim f_n(x)$ 存在 for $\forall x \in X$

(a) $\exists$ 开 $V \neq \emptyset$，$M < \infty$ s.t. $|f_n(x)| < M$ on V

(b) $\forall \varepsilon > 0$，$\exists$ 开 $V \neq \emptyset$，$N \in \mathbb{N}^*$，$|f(x) - f_n(x)| \le \varepsilon$ for $n > N$，$x \in V$

证：(a) $A_M = \{x : |f_n(x)| \le M$ for all $n\}$ 则 $X = \bigcup_{M \in \mathbb{N}} A_M$

$\therefore \exists M$，A_M 有内点

(b) $A_N = \{x : |f_n(x) - f_m(x)| \le \varepsilon$ for $n, m > N\}$ $X = \bigcup A_N$

$\therefore \exists N$，A_N 有内点

9. 设 $\Phi \subset L^1(X)$ 一致可积(UI)，若 $\forall \varepsilon>0$，$\exists \delta>0$

$\left|\int_E f\,dx\right|<\varepsilon$ ~~$\int_E |f|\,dx<\varepsilon$~~ if $f\in\Phi$，$\mu(E)<\delta$

Vitali Convergence Thm VCT

(a) $\mu(A)<\infty$，$\{f_n\}$ UI，$f_n\to f$ ae，f

则 $\int_A |f_n-f|\,dx = 0$

注：以上条件同 $\int_E |f|\,dx<\varepsilon$ 等价

$E\cap\{f>0\}$

$E\cap\{f<0\}$

证：我们有：$\int_A |f_n-f|\,dx \le \int_{\underbrace{\{|f_n-f|>\varepsilon\}}_{A_n(\varepsilon)}} |f_n-f|\,dx + \varepsilon\, m(A)$

而 $f_n\to f$ a.e $\Rightarrow$ $f_n\to f$ in measure $\therefore m(A_n(\varepsilon))\to 0$

且 $\int_{A_n(\varepsilon)} |f_n-f|\,dx \le \int_{A_n(\varepsilon)} |f_n|+|f|\,dx$

$\le \liminf\limits_{m\to\infty} \int_{A_n(\varepsilon)} |f_n|+|f_m|\,dx \to 0$ as $n\to\infty$

(由UI)

(b) $L^1(A)$中有限集合必是UI ~ 显然

(c) $\mu(X)<\infty$，$f_n\in L^1(X)$，

$\lim\limits_{n\to\infty}\int_E f_n\,dx$ 存在，for $\forall$可测E

则 $\{f_n\}$ UI $\qquad \mathcal{L}=\{B\subset A: B可测\}$

证：定义 $d(A,B)=\int |\chi_A-\chi_B|\,dx$，则 $(\mathcal{L}, d)$ 为度量空间且完备（L^1 完备）

$E\to \int_E f_n\,dx$ ~~$\int_E |f_n|\,dx$~~ 连续 for each n

由前，$\exists E_0$，δ，N s.t.

$\left|\int_E f_n - f_N\,dx\right| < \varepsilon$ if $d(E,E_0)<\delta$，$n>N$

若 $\mu(A)<\delta$，则 $E=E_0-A$，$E=E_0\cup A$ 时上式成立

$A=(E_0\cup A)-(E_0-A)$ $\quad \left|\int_A (f_n-f_N)\,dx\right| = \left|\int_{E_0\cup A} - \int_{E_0-A}\,dx\right| < 2\varepsilon$

而 $\{f_1,\cdots,f_N\}$ UI

$\therefore \exists\delta'$，$\mu(A)<\delta' \Rightarrow \left|\int_A f_n\,dx\right|<3\varepsilon$

P 7-1.2

$\overline{\Phi}(f) = \underline{\Phi}(f)$ a.e $\Leftrightarrow$ f is continuous a.e

$$\omega_f(x_0) = \lim_{\delta\to 0} \sup\{|f(x') - f(x'')| : x', x'' \in B(x_0, \delta)\}$$

$\omega_f(x_0)$ 为 f 在 x_0 处的振幅

$$\Delta^{(n)}: a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{k_n}^{(n)} = b \quad (n=1,2,\cdots)$$

$$|\Delta^{(n)}| = \max\{x_i^{(n)} - x_{i-1}^{(n)} : 1\le i\le k_n\} \qquad \lim_{n\to\infty}|\Delta^{(n)}| = 0$$

$$M_i^{(n)} = \sup\{f(x) : x_{i-1}^{(n)} \le x \le x_i^{(n)}\}$$

$$m_i^{(n)} = \inf\{f(x) : x_{i-1}^{(n)} \le x \le x_i^{(n)}\}$$

$$\overline{\int_a^b} f(x)dx \triangleq \lim_{n\to\infty}\sum_{i=1}^{k_n} M_i^{(n)}(x_i^{(n)} - x_{i-1}^{(n)}) \qquad \underline{\int_a^b} f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{k_n} m_i^{(n)}(x_i^{(n)} - x_{i-1}^{(n)})$$

Lemma: $\int_I \omega(x)\,dx = \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx$.

(Lebesgue 积分)

$f(x)$ 在 $[a,b]$ 上有界 $\omega(x)$ 有界 $\omega(x)$ 可测 P38. 阅读强.

~~$\omega \in L[a,b]$~~ ω Lebesgue 可积

$$\omega_{\Delta^{(n)}}(x) = \begin{cases} M_i^{(n)} - m_i^{(n)} & x\in(x_{i-1}^{(n)}, x_i^{(n)}), \\ 0 & \end{cases}$$

分点.

m(分点集合) $= 0$ 且由 $\omega(x)$ 定义知. $\lim_{n\to\infty}\omega_{\Delta^{(n)}}(x) = \omega(x)$.

$$\omega_{\Delta^{(n)}}(x) \le \max_{[a,b]} f(x) - \min_{[a,b]} f(x)$$

由 DCT. $\lim_{n\to\infty}\int_I \omega_{\Delta^{(n)}}(x)dx = \int_I \omega(x)dx$

$$= \sum_{i=1}^{k_n}(M_i^{(n)} - m_i^{(n)})(x_i^{(n)} - x_{i-1}^{(n)}) = \sum_{i=1}^{k_n} M_i^{(n)}(x_i^{(n)} - x_{i-1}^{(n)}) - \sum_{i=1}^{k_n} m_i^{(n)}(x_i^{(n)} - x_{i-1}^{(n)})$$

$$\Rightarrow \int_I \omega(x)dx = \cancel{\lim_{n\to\infty}} \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx.$$

∴ 若 f Riemann 可积 ⇒ 上下和相等 ⇒ $\int \omega(x)dx=0 \Rightarrow \omega(x)=0$ a.e.

f(x) 几乎处处连续 ②

· ⇐ m(不连续点)$=0 \Rightarrow \int \omega(x)dx=0=\overline{\int_a^b} f(x)dx-\underline{\int_a^b} f(x)dx \Rightarrow$ Riemann 可积

注 Riemann 可积 ⟹ Lebesgue 可积

P 7-2-3 Let $L: \mathbb{R}^d \to \mathbb{R}^d$ be a linear transform

prove: For any measurable set $A \subset \mathbb{R}^d$ $m(L(A))=|\det L|m(A)$

对角阵 对角元为1 下三角 当L不满秩

$L=L_1 \Delta L_2$ $|\det L|=0$ 此时等式自然成立

对角元为1.上三角

先证 $m(L_1(E))=|\det L_1|m(E)=m(E)$

$d=2$ $\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$。

$\chi_{L_1(E)}(x,y)=\chi_E(L^{-1}(x,y))=\chi_E(x-ay,y)$ 平移不变性

$m(L_1(E))=m(E)$?

$d=n$ ⟹ $\chi_{L_1(E)}(x_1,\cdots x_n)=\chi_E(L_1^{-1}(x_1,\cdots x_n))$ L_1^{-1} 也为上三角阵

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 1 & \cdots & a_{2n} \\ & & \ddots & \\ & & & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

$\cancel{x_i=}\ x_i=x_i'+\sum_{d=i+1}^{n} a_{id}x_d'$

$$\underbrace{\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{n个} \chi\left(x_1+\sum_{d=2}^{n}a_{1d}x_d',\ x_2+\sum_{d=3}^{n}a_{2d}x_d' \cdots\right) dx_1dx_2\cdots dx_n$$

$$=\underbrace{\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{n个} \chi(x_1,x_2,\cdots x_n)dx_1\cdots dx_n$$

$=m(E)$

$\Delta=\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}$ 开长方体

设 $\bigcup O_i$ 为 E 的一个开覆盖 $\bigcup_{i=1}^{\infty} O_i\Delta$ 为 EΔ 的覆盖 $\sum |O_i| \le m(E)+\varepsilon$

$m(\Delta O_i)=|\lambda_1\cdots\lambda_d|m(O_i)=\cancel{|\lambda_1\cdots\lambda_d|m(E)}$

$m(\Delta E)\le m(\Delta \bigcup O_i)\le|\lambda_1\cdots\lambda_d|\sum m(O_i)\le\lambda_1\cdots\lambda_d(m(E)+\varepsilon)$

对于 Δ^{-1} $m(\Delta E)\le|\det\Delta|m(E)$ $m(\Delta^{-1}\Delta E)\le|\det\Delta^{-1}|m(\Delta E)$ ∴ $m(\Delta E)\ge|\det\Delta|m(E)$

⑧

$\lim_{p\to p_0^-}\|f\|_{L^p}=\|f\|_{L^{p_0}}$, $\forall 0<p_0<+\infty$ 单调收敛

$0<p_1<p_2<\infty$ $\|f\|_{L^{p_1}}\le(\mu(X))^{\frac{1}{p_1}-\frac{1}{p_2}}\|f\|_{L^{p_2}}$.

(2) $\int_E|f|^{p_1}d\mu=\int_E|f|^{p_1}\cdot 1\,d\mu\le\left(\int_E|f|^{p_1\cdot p}d\mu\right)^{\frac{1}{p}}\left(\int_E 1^q d\mu\right)^{\frac{1}{q}}$

令 $p=\frac{p_2}{p_1}>1$ $q=\frac{1}{1-\frac{1}{p}}=\frac{1}{1-\frac{p_1}{p_2}}=\frac{p_2}{p_2-p_1}$

$=\left(\int_E|f|^{p_2}d\mu\right)^{\frac{1}{p_2}\cdot p_1}\cdot m(E)^{\frac{p_2-p_1}{p_2}}$

$\left(\int_E|f|^{p_1}d\mu\right)^{\frac{1}{p_1}}\le\left(\int_E|f|^{p_2}d\mu\right)^{\frac{1}{p_2}}\cdot m(E)^{\frac{p_2-p_1}{p_1p_2}}$ $\frac{p_2-p_1}{p_1p_2}=\frac{1}{p_1}-\frac{1}{p_2}$

$\therefore$ $L^{p_2}(X)\subset L^{p_1}(X)$

$f(x)=\frac{1}{x}$ $f\in L^{\frac{1}{2}}[0,1]$ $f\notin L^1[0,1]$

③

① 反Hölder不等式

$\boxed{0<p<1,\ q<0}$ $\frac{1}{p}+\frac{1}{q}=1$. 对 $f\in L^p(E)$. $g\in L^q(E)$ 有

$\int_E |f(x)g(x)|dx \geq \|f\|_p\|g\|_q$.

Proof: ~~不妨设~~ $fg\in L(E)$ $\boxed{\frac{1}{p}>1}$ $\frac{1}{1-p}>1$

$$\|f\|_p=\left(\int_E \frac{|f(x)g(x)|^p}{|g(x)|^p}dx\right)^{\frac{1}{p}} \leq \left(\int_E |f(x)g(x)|^{p\cdot\frac{1}{p}}\right)^{\frac{p}{p}}\left(\int\left[\frac{1}{|g(x)|^p}\right]^{\frac{1}{1-p}}dx\right)^{\frac{1-p}{p}}$$

$p'=\frac{1}{p}>1$ $q'=\frac{1}{1-p}>1$ $\frac{1}{p'}+\frac{1}{q'}=1$ $q=\frac{p}{p-1}$

$$Q=\left(\int_E |f(x)g(x)|^{p\cdot p'}\right)^{\frac{1}{pp'}}\left(\int\left[\frac{1}{|g(x)|^p}\right]^{q'}\right)^{\frac{1}{pq'}} \quad -q.$$

$=\frac{\|fg\|_{L^1}}{\|g\|_q}$

$g(x)^{-pq'}=g(x)^{\frac{-p}{1-p}}=g(x)^{\frac{p}{p-1}}=g(x)^q$

② $0<p<1$

反Minkowski不等式. $0<p<1$. $f,g\in L^p(E)$ 有 $q=\frac{p}{p-1}$

$\||f|+|g|\|_p \geq \|f\|_p+\|g\|_p$.

$$\||f|+|g|\|_p^p=\int_E(|f|+|g|)^{p-1}(|f|+|g|)dx \overset{\text{反Hölder}}{\geq} \left(\int_E(|f(x)|+|g(x)|)^{q(p-1)}dx\right)^{\frac{1}{q}}(\|f\|_p+\|g\|_p)$$

$= \||f|+|g|\|_p^{p-\frac{p}{q}}$

$\||f|+|g|\|_p^{\frac{p}{q}}\left(\int_E(|f|+|g|)^{p}dx\right)^{\frac{1}{p}\cdot\frac{p}{q}}(\|f\|_p+\|g\|_p)$

$\geq \|f\|_p+\|g\|_p$. $\frac{1}{q}+\frac{1}{p}=1$

$\|f+g\|_{L^p}^p \le \|f\|_{L^p}^p + \|g\|_{L^p}^p$

$\|f+g\|_{L^p}^p = \left(\int |f+g|^p dx\right)^{\not p} \le \int (|f|+|g|)^p dx$. ④

下证 $(|f|+|g|)^p \le |f|^p+|g|^p$ $0<p<1$ 时.

不妨设 $|f|>0$ $|g|>0$. 否则很显然

下证 $\left(\frac{|f|}{|f|+|g|}\right)^p + \left(\frac{|g|}{|f|+|g|}\right)^p \ge 1$

即 $t^p+(1-t)^p \ge 1$ $(0<p<1)$

$f'(t) = pt^{p-1} - p(1-t)^{p-1} = 0$ $t^{p-1}=(1-t)^{p-1}$ $\left(\frac{1}{t}-1\right)^{p-1}=1$.

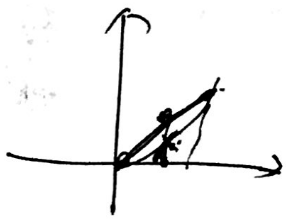
$\frac{1}{t}-1=1$ $t=\frac{1}{2}$


$\left(\frac{1}{2}\right)^p+\left(\frac{1}{2}\right)^p \ge 1$ $0<p<1$ ✓

$\therefore (|f|+|g|)^p \le |f|^p+|g|^p$.

$\therefore \|f+g\|_{L^p}^p \le \int |f|^p+|g|^p dx = \|f\|_{L^p}^p + \|g\|_{L^p}^p$

$0<p<1 \Rightarrow \frac{1}{p}>1$. $\||f|+|g|\|_{L^p} \le 2^{\frac{1}{p}-1}(\|f\|_{L^p}+\|g\|_{L^p})$.



$\left(\frac{a+b}{2}\right)^{\frac{1}{p}} \le \frac{a^{\frac{1}{p}}+b^{\frac{1}{p}}}{2}$ $a,b \ge 0$.

$\le \left(\int (|f|^p+|g|^p) dx\right)^{\frac{1}{p}} \le \frac{\left(\int_E |f|^p dx\right)^{\frac{1}{p}} + \left(\int_E |g|^p dx\right)^{\frac{1}{p}}}{2} \times 2^{\frac{1}{p}}$

$\left[\int (|f|+|g|)^p dx\right]^{\frac{1}{p}}$

$= 2^{\frac{1}{p}-1}\left[\left(\int_E |f|^p dx\right)^{\frac{1}{p}} + \left(\int_E |g| dx\right)^{\frac{1}{p}}\right]$

$= \|f+g\|_{L^p}$.

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补充:

1. 记 $L^2\cdot L^2=\{f\cdot g \mid f\in L^2, g\in L^2\}$ 证明: $L^1=L^2\cdot L^2$.

⑥

证: $f,g\in L^2$. $\|fg\|_{L^1}\le \|f\|_{L^2}\|g\|_{L^2} \Rightarrow L^2\cdot L^2\subset L^1$

反之 取 $\varphi\in L_1$. 取 $f=\operatorname{sgn}(\varphi)\sqrt{|\varphi|}$. $g=\sqrt{|\varphi|}$. $\varphi=fg$. $f,g\in L^2$.

说明 $L^1\subset L^2\cdot L^2$

$L^1\subset L^p\cdot L^q$. $(1=\frac{1}{p}+\frac{1}{q})$ (推广) $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}\Rightarrow L^r=L^p\cdot L^q$

2. $1<p,q<\infty$ $\frac{1}{p}+\frac{1}{q}=1$, $f\in L^p$. $g\in L^q$. 证明 $f*g\in C_0$.

连续且 $\lim\limits_{x\to\infty}f(x)=0$.

$$f*g(x)=\int f(x-t)g(t)dt$$

$$|f*g(x+h)-f*g(x)|\le\int|f(x+h-t)-f(x-t)||g(t)|dt$$

$$\le\left(\int|f(x+h-t)-f(x-t)|^p dt\right)^{\frac{1}{p}}\|g\|_{L^q}=\left(\int|f(h-t)-f(-t)|^p dt\right)^{\frac{1}{p}}\|g\|_{L^q}$$

$\downarrow 0.$ 有界.

$\forall\varepsilon>0$ $\exists R$ s.t $\int_{B(0,R)^c}|f|^p<\varepsilon^p$ $\int_{B(0,R)^c}|g|^q<\varepsilon^q$

当 $x>2R$ 时

$$|f*g(x)|\le\int_{B(0,R)}|f(x-t)g(t)|dt+\int_{B(0,R)^c}|f(x-t)g(t)|dt$$

$$\le\left(\int_{B(0,R)}|f(x-t)|^p dt\right)^{\frac{1}{p}}\left(\int_{B(0,R)}|g(t)|^q dt\right)^{\frac{1}{q}}+\left(\int_{B(0,R)^c}|f(x-t)|^p dt\right)^{\frac{1}{p}}\left(\int_{B(0,R)^c}|g(t)|^q\right)^{\frac{1}{q}}$$

$$\le\int_{B(x,R)}|f(-t)|^p dt\,\|g\|_{L^q}+\|f\|_{L^p}\cdot\varepsilon$$

$$=\int_{B(-x,R)}|f(t)|^p dt\,\|g\|_{L^q}+\|f\|_{L^p}\cdot\varepsilon$$

$$<(\|g\|_{L^q}+\|f\|_{L^p})\,\varepsilon\to 0.$$

当 $x>2R$ $B(-x,R)\cap B(0,R)=\varnothing$