3. Prove the following ...

PS13 P_1

1、Let μ be a signed measure on $(X,\mathcal{F})$. prove:

(1) If $A_n\in\mathcal{F}$, and $A_1\subset A_2\subset\cdots$, then $\mu(\bigcup_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n)$

准备工作: $\cancel{\mu(\bigcup A_n)=\mu\{\bigcup_{n=1}^{\infty}(A_n\cap A)}$ Hahn μ is a signed measure $\Rightarrow$

$X=X_+\cup X_-$, positive, negative, $X_+\cap X_-=\phi$

Jordan decomposition: μ signed measure $\Rightarrow$ $\exists$ unique $\mu=\mu_+-\mu_-$ s.t.

s.t. $\cancel{\mu_+|_{X_+}}=\mu_+|_{X_-}=0$, $\mu_-|_{X_+}=0$. μ_+、μ_- 中至少一个在测度空间内有限.

proof: $\mu(\bigcup_{n=1}^{\infty}A_n)=\mu\{\bigcup_{n=1}^{\infty}(A_n\cap X_+)\cup(A_n\cap X_-)\}$

$=\mu\{(\bigcup_{n=1}^{\infty}A_n\cap X_+)\cup(\bigcup_{n=1}^{\infty}A_n\cap X_-)\}=\mu_+[\bigcup_{n=1}^{\infty}A_n\cap X_+]-\mu_-[\bigcup_{n=1}^{\infty}A_n\cap X_-]$

$A_1\cap X_+\subset A_2\cap X_+\subset\cdots$

$A_1\cap X_-\subset A_2\cap X_-\subset\cdots$

$\therefore \mu(\bigcup_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu_+(A_n\cap X_+)-\lim_{n\to\infty}\mu_-(A_n\cap X_-)$

$=\lim_{n\to\infty}[\mu_+(A_n)-\mu_-(A_n)]$

$=\lim_{n\to\infty}\mu(A_n)$.

(2) $|\mu(A_1)|<+\infty$ $\therefore|\mu_+(A_1)-\mu_-(A_1)|<+\infty$

$\because\mu_+$ 与 μ_- 中最多只有一个为 $+\infty$ $\therefore$ $\mu_+(A_1)<\infty$ $\mu_-(A_1)<\infty$

$\mu(\bigcap_{n=1}^{\infty}A_n)=\mu[((\bigcap_{n=1}^{\infty}A_n)\cap X_+)\cup(\bigcap_{n=1}^{\infty}A_n\cap X_-)]=\mu_+((\bigcap_{n=1}^{\infty}A_n)\cap X_+)-\mu_-((\bigcap_{n=1}^{\infty}A_n)\cap X_-)$

$A_1\cap X_+\supset A_2\cap X_+\cdots$

$A_1\cap X_-\supset A_2\cap X_-\cdots$

$B_i^+=A_1\cap X_+\setminus(A_i\cap X_+)$, $B_{i-1}^+\subset B_i^+\cdots$, $B_1^+=\phi$,

$\mu_+(\bigcup_{i=1}^{\infty}B_i^+)=\lim_{n\to\infty}\mu_+(B_i)$

$\mu_+(A_1\cap X_+)-\mu_+(\bigcap_{n=1}^{\infty}A_n\cap X_+)=\mu_+(A_1\cap X_+)-\lim_{n\to\infty}(A_n\cap X_+)$

$\therefore$ 原式 $=\lim_{n\to\infty}\mu_+(A_n\cap X_+)-\mu_-(A_n\cap X_-)$, $\mu_-(A_n\cap X_+)=0$, $\mu_+(A_n\cap X_-)=0$

$=\lim_{n\to\infty}[\mu_+(A_n)-\mu_-(A_n)]=\lim_{n\to\infty}\mu(A_n)$

① μ, ν 右⇒左 signed measures prove $\mu \perp \nu \Leftrightarrow |\mu| \perp \nu$

(1) $\Leftarrow$ $\exists X_1 \cup X_2 = X$ $X_1 \cap X_2 = \emptyset$. $|\mu||_{X_2} = 0$ $\nu|_{X_1} = 0$.

$\mu_+|_{X_2} = 0$ $\mu_-|_{X_2} = 0$ $\nu|_{X_1} = 0 \Rightarrow \mu|_{X_2} = 0$, $\nu|_{X_1} = 0. \Rightarrow$ 左

左⇒右.

"⇒" ∵ $\mu \perp \nu$ ∴ $\exists X_2$ s.t $\mu|_{X_2} = 0$. 但若 $|\mu||_{X_2} \neq 0$ 则 $\exists A \subset X_2$.

s.t $|\mu|(A) > 0$ ~~假设 $\mu_+(A \cap X_2) > 0$. 令 $B = A \cap X_2 \subset A \subset X$.~~
把 $X = X_+ \cup X_-$ s.t $\mu_+|_{X_-} = 0$ $\mu_-|_{X_+} = 0$

~~$\mu(B) = \mu(A \cap X_2)$~~ 不妨假设 $\mu_+(A \cap X_+) > 0$. 取 $B = A \cap X_+$

$A \cap X_+ \subset A \subset X_2$ $\mu(B) = \mu(A \cap X_+) = \mu_+(A \cap X_+) > 0$ $\mu_-(A \cap X_+) = 0$

由此则与 $\mu|_{X_2} = 0$ 矛盾. ∴ $|\mu||_{X_2} = 0 \Rightarrow |\mu| \perp \nu$.

(2) $\mu \ll \nu \Leftrightarrow |\mu| \ll \nu$
$\mu \ll \nu \Leftrightarrow |\mu| \ll \nu$

$\nu \ll \mu$ ~~if and only if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t $\forall A \in \mathcal{f}$~~

~~if $\mu(A) < \delta$ then $|\nu(A)| < \varepsilon$.~~ if every μ-null set is also a ν null-set.

"⇐" 显然

"⇒" ~~若 $|\mu| \ll \nu$ 不成立 $\exists \varepsilon > 0$ $\forall \delta$~~ st $\exists A$ ~~if $\nu(A) < \delta$ 但 $|\mu|(A) \geq \varepsilon$~~

~~$\mu_+(A)$ $\mu_+(A \cap X_+) + \mu_-(A \cap X_-) > \varepsilon$~~ ~~$A \cap X_+ \subset A$~~

$\exists$ a null set. $\nu(A) = 0$ $\mu(A) = 0$ 但 $|\mu|(A) \neq 0$. ~~不妨设 $\mu_+(A) > 0$~~

即 ~~$\mu(A \cap X_+) \geq 0$~~. $\mu_+(A \cap X_+) + \mu_-(A \cap X_-) > 0$. 不妨设 $\mu_+(A \cap X_+) > 0$
$A \cap X_+ \subset A$

$\mu(A \cap X_+) = \mu_+(A \cap X_+) - \mu_-(A \cap X_+) > 0$. 矛盾.

(3) $\nu_1, \nu_2 \ldots$ a sequence of (positive) measures : $\nu_j \perp \mu \ \forall j \Rightarrow \sum_{j=1}^{\infty} \nu_j \perp \mu$.

(3) $\nu_1 \ldots$ ν_2. positive measures

$\nu_j \perp \mu \Leftrightarrow \nu_j \perp |\mu|$, $\sum_{j=1}^{\infty} \nu_j \perp \mu \Leftrightarrow \sum_{j=1}^{\infty} \nu_j \perp |\mu|$

$\forall j$ $\nu_j \perp |\mu| \Rightarrow \sum_{j=1}^{\infty} \nu_j \perp |\mu| \Rightarrow \sum_{j=1}^{\infty} \nu_j \perp \mu$.

~~$\forall A \in \mathcal{f}$~~ $\exists M_{j1}$ M_{j2} s.t ν_j ~~(M_{j1})~~ $\nu_j|_{M_{j1}} = 0$

$|\mu||_{M_{j2}} = 0$ 令 $M_2 = \bigcup_{j=1}^{\infty} M_{j2}$. $M_1 = X \setminus M_2$

$\sum_{j=1}^{\infty} \nu_j(M_1) = \sum_{j=1}^{\infty} \nu_j(\cap (X \setminus M_{j2})) \leq \sum_{j=1}^{\infty} \nu_j(X \setminus M_{j2}) = 0$.
$= M_{j1}$

$|\mu|(M_2) \leq \sum_{j=1}^{\infty} |\mu|(M_{j2}) = 0$. ∴ $\sum_{j=1}^{\infty} \nu_j \perp |\mu|$

(4) $\nu_j \ll \mu$, $\forall j$ $\sum_{j=1}^{\infty} \nu_j \ll \mu$

$\forall A \in \mathcal{F}$. $\mu(A)=0$. 则 $\nu_j(A)=0$. $\forall j$. $\underline{\forall B \subset A}$ s.t $\nu_j(B) \le \nu_j(A)=0$

$\sum_{j=1}^{\infty} \nu_j(B)=0$ 故. A 为 $\sum_{j=1}^{\infty} \nu_j$ 的 Null set.

$\Rightarrow \sum_{j=1}^{\infty} \nu_j \ll \mu$.

3. ν、μ、ρ. finite positive measures on $(X, \mathcal{F})$, suppose $\nu \le \mu \le \rho$.

(1) Let $f \in L^1(X, \nu)$. Prove: $f\frac{d\nu}{d\mu} \in L^1(X, \mu)$ and $\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu$

$\nu \le \mu$ positive measure. 不妨设 ~~f>0~~ ν positive. $\nu = \mu_g$ ~~(∃g measurable)~~.

∃ g 非负. $\nu(A) = \int_A g \, d\mu$ ($\nu = \mu_g$) $\frac{d\nu}{d\mu} = g$. $\forall A \in \mathcal{F}$.

∃ ~~extended integrable~~ $g \ge 0$ s.t $\nu = \mu_g$. $d\nu = g \, d\mu \Leftrightarrow g = \frac{d\nu}{d\mu}$

先证 $\int_X f \, d\nu = \int_X f g \, d\mu$.

① 若 f 为 simple function. $f = \sum_{j=1}^{n} c_j \chi_{A_j}(x)$ $\sum_{j=1}^{n} c_j \int_X \chi_{A_j}(x) d\nu$

$= \sum_{j=1}^{n} c_j \nu(A_j) = \sum_{j=1}^{n} c_j \int_{A_j} g \, d\mu$.

② f 非负可测 $f_n \le f_{n+1} \le \cdots \to f$. $f_n g \to f g \ge 0$.

$\int_X f d\nu = \lim_{n\to\infty} \int_X f_n d\nu = \lim_{n\to\infty} \int_X f_n g \, d\mu = \int_X f g \, d\mu$.

③ 一般. $\int_X f^+ d\nu = \int_X f^+ g \, d\mu$. $\int_X f^- d\nu = \int_X f^- g \, d\mu$ $f \in L^1(X, \nu) \Rightarrow \therefore fg \in L^1(X,\mu)$

$\int_X (fg)^+ d\mu = \int_X f^+ g \, d\mu < +\infty$ $\int_X (fg)^- d\mu = \int_X f^- g \, d\mu < +\infty$

$\int_X f d\nu = \int_X f g \, d\mu$ $g = \frac{d\nu}{d\mu}$.

(2) $\nu \ll \rho$ 易得. $\int_X f d\nu = \int_X f \frac{d\nu}{d\rho} d\rho$ $\nu \ll \rho$.

$\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu = \int_X$ ~~$f\frac{d\nu}{d\mu}$~~ ($\mu \le \rho$)

$(\nu \le \mu)$ $\int_X f \frac{d\nu}{d\mu} \frac{d\mu}{d\rho} d\rho$.

$\int_X f d\nu = \int f \frac{d\nu}{d\rho} d\rho = \int_X f \frac{d\nu}{d\mu} \frac{d\mu}{d\rho} d\rho$ $\forall f$.

取 $f = \chi_A$ $\int_A \left(\frac{d\nu}{d\mu} \frac{d\mu}{d\rho} - \frac{d\nu}{d\rho} \right) d\rho = 0$. $\forall A \in \mathcal{F}$

$\therefore \frac{d\nu}{d\rho} = \frac{d\nu}{d\mu} \frac{d\mu}{d\rho}$ ρ.a.e.

prop: Let ν be a finite signed measure, and let μ be a (positive) measure on $(X, \mathcal{F})$. Then $\nu \ll \mu$ if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall A \in \mathcal{F}$, if $\mu(A) < \delta$, then $|\nu(A)| < \varepsilon$

"$\Leftarrow$" 若 A 为 μ 的 null set 下证 $\forall B \subset A$ $\nu(B) = 0$

~~$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.~~ $\mu(B) = 0 < \delta$ $|\nu(B)| < \varepsilon$ $\forall \varepsilon > 0$

令 $\varepsilon \to 0$ $\nu(B) = 0$. #

"$\Rightarrow$" 反证 $\nu \ll \mu$.

但 $\exists \varepsilon_0 > 0$ $\forall n$. $\exists A_n \in \mathcal{F}$ $\mu(A_n) < \frac{1}{2^n}$ 但 $|\nu(A_n)| \geq \varepsilon_0$.

$A^* = \overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ $\quad \sum_{n=1}^{\infty} \mu(A_n) < +\infty$ $\therefore \mu(A^*) = 0$

$\nu \ll \mu$. $\Rightarrow \nu(A^*) = 0$.

ν^+、ν^- 均有限 $\quad \lim_{n\to\infty} |\nu| \left(\bigcup_{k=n}^{\infty} A_k\right) = |\nu|\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 0$

但 $|\nu|\left(\bigcup_{k=n}^{\infty} A_n\right) \geq |\nu(A_n)| \geq \varepsilon_0$. 矛盾.

a

$\vdash 0$

ν

$\geq \varepsilon$

≥ 0

≥ 0.

> 0

2 prove the following ...

PSB. P2.

1. prove the Lebesgue differential theorem ~~...~~.

Let $f \in L^1_{loc}(\mathbb{R}^d)$. Then for a.e. $x \in \mathbb{R}^d$. one has

(1) $\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(t)\,dt = f(x)$

(2) $\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)-f(x)|\,dt = 0$

proof: Stein. P104.

$E_\alpha = \{x.\ \overline{\lim_{r\to 0}} \left|\frac{1}{m(B(x,r))}\int_{B(x,r)} f(t)\,dt - f(x)\right| > 2\alpha\}$

~~要证~~ $m(E_\alpha)=0$

这样保证了 $m(E) = m(\bigcup_{n=1}^{\infty} E_{\frac{1}{n}}) = 0$.

fix α. $\exists g$ s.t $\|f-g\|_{L^1(\mathbb{R})} < \varepsilon$ g为紧支集连续函数

$\because$ g连续 $\therefore$ $\lim_{r\to 0} \frac{1}{m(B)} \int_B g(y)\,dy = g(x)$

$\therefore \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)-f(x)|\,dt \le \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)-g(t)| + |g(t)-g(x)| + |g(x)-f(x)|\,dt =$

$\le (f-g)^*(x). + |g(x)-f(x)|.$

$F_\alpha: \{x: (f-g)^*(x) > \alpha\}$ $\quad G_\alpha = \{x: |f-g| > \alpha\}$

$m(G_\alpha) \le \frac{1}{\alpha}\|f-g\|_{L^1} \le \frac{\varepsilon}{\alpha}$

$m(F_\alpha) \le \frac{A}{\alpha}\|f-g\|_{L^1} = \frac{A}{\alpha}\varepsilon$

$\therefore \overline{\lim} \le (f-g)^*(x) + |f(x)-g(x)|$ $\quad E_\alpha \subset F_\alpha \cup G_\alpha$

令 $\varepsilon \to 0$ $m(E_\alpha) = 0.$

当 $x \in \mathbb{R}^d \setminus E_\alpha$ 时

自然成立 $\therefore (2)$ 成立 $\Rightarrow (1)$.

2、$f \in L^1(\mathbb{R}^d)$.

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)|\,dt$$

(1) Mf is measurable.

(2) $Mf(x) < +\infty$ for a.e. $x \in \mathbb{R}^d$

(3) $Mf(x) \geq |f(x)|$. a.e. $x \in \mathbb{R}^d$

(4) suppose f is not identically zero. prove: Mf is not integrable.

(1) 设 $E_\alpha = \{x \in \mathbb{R}^d \ Mf(x) > \alpha\}$ 要证它是开集

$\forall x_0 \in E_\alpha$. $Mf(x_0) > \alpha$. $\exists r$ s.t $\frac{1}{m(B(x_0,r))} \int_{B(x_0,r)} |f(y)|\,dy = t_1 > \alpha$

$\forall x \in B(x_0, \delta)$ $B(x_0,r) \subset B(x, r+\delta)$

$$\int_{B(x,r+\delta)} |f(y)|\,dy \geq \int_{B(x_0,r)} |f(y)|\,dy = t_1\, m(B(x_0,r))$$

$\exists \delta$ s.t $(r+\delta)^d < \frac{t_1}{\alpha} r^d$

$$= t_1 \left(\frac{r}{r+\delta}\right)^d m(B(x,r+\delta)) > \alpha\, m(B(x,r+\delta))$$

$\therefore$ ~~$Mf(x)$~~ $Mf(x) \geq \frac{1}{m(B(x,r+\delta))} \int_{B(x,r+\delta)} |f(y)|\,dy > \alpha$

$\forall x \in B(x_0, \delta)$

$\therefore$ 开集

(2)

$$m(x \in \mathbb{R}^d.\ Mf(x) > \alpha) \leq \frac{3^d \|f\|_1}{\alpha}$$

$m(x \in \mathbb{R}^d : Mf(x) = +\infty)\} = 0$. $\therefore < +\infty$ a.e

(3) $\left| \frac{1}{m(B(x,r))} \int_{B(x,r)} f(t)\,dt \right| \leq \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)|\,dt \leq Mf(x)$

由1、知.

~~左~~ $\lim_{r\to 0}$ 左 $= f(x)$. a.e. $\therefore |f(x)| \leq Mf(x)$ a.e.

(4) $Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f|\,dx$

不妨设 在$B(x,r_0)$上 $|f| > \varepsilon$ 固定

$Mf(x) \geq \frac{1}{B(x,|x|+r_0)} \int_{B(0,r_0)} |f(t)| \geq \frac{\varepsilon r_0^d}{(x+r_0)^d}$ 固定

$\int Mf(x) \geq \varepsilon r_0^d \int \frac{1}{(|x|+r_0)^d}\,dx$

$\sim \frac{1}{|x|^d}$ 在$\mathbb{R}^d$中不可积

3. prove the following vitali covering lemma:

Let $A \subset \mathbb{R}^d$ be a measurable set with $m(A) < \infty$

Let $\mathcal{U} = \{B(x_\alpha, r_\alpha)\}$ be a vitali covering of A. 记.

$\forall x \in A, \forall r > 0$ $\exists B \in \mathcal{U}$ s.t $x \in B$ 且 radius $B < r$

prove: one can find fintely many balls $B_1 \cdots B_N$ in $\mathcal{U}$ that are pairewise disjoint s.t. $\sum_{i=1}^{N} m(B_i) \geq m(A) - \varepsilon$

证: Lemma. $U \subset \bigcup_\alpha B(x_\alpha, r_\alpha)$. $\forall c < m(U)$. $\exists$ fintely many pairwise disjoint balls $B_j = B(x_j, r_j)$ in the family of $\mathcal{U}$, s.t $\sum_{j=1}^{n} m(B_j) > 3^{-d}\varepsilon$

~~$\exists B_1, \cdots B_{N_1}$ 两两不交 $\sum_{k=1}$~~ $0 < \varepsilon < m(A)$

$\exists B_1, \cdots B_{N_1}$ 两两不交. $\sum_{k=1}^{N_1} m(B_k) > 3^{-d}\varepsilon$

若 $\sum_{k=1}^{N_1} m(B_k) \geq m(A) - \varepsilon$ 则证毕.

否则 设 $A_1 = A \setminus \left(\bigcup_{k=1}^{N_1} \overline{B_k}\right)$ $\forall x \in A_1$. $d(x, \bigcup_{k=1}^{N_1} \overline{B_k}) = r > 0$

$\exists B$ s.t $x \in B$ $B \cap \bigcup_{k=1}^{N_1} \overline{B_k} = \emptyset$

$\therefore$ 类似构造: $\exists B_{N_1+1} \cdots B_{N_2}$ 两两不交

$\sum_{k=N_1+1}^{N_2} m(B_k) > 3^{-d}\varepsilon$. ...... 若 $\sum_{k=1}^{N_2} m(B_k) \geq m(A) - \varepsilon$ #

否则 $\exists N$ s.t. N_N. $N \cdot 3^{-d}\varepsilon \geq m(A) - \varepsilon$

#

4. Let $f: \mathbb{R}^d \to [0, +\infty]$ be a measurable function. consider the Borel measure $d\mu = f dx$.

prove: $f \in L^1_{loc}(\mathbb{R}^d) \Leftrightarrow \mu$ is locally finite and outer regular

μ局部有限: $\forall K \subset X$ K compact $\mu(K) < +\infty$

$f \in L_{loc}(\mathbb{R}^d)$ $\forall B(x, r)$ $\int_{B(x,r)} f dx < +\infty$.

"$\Leftarrow$" $\int_{B(x,r)} |f| dx \leq \mu(\overline{B}(x,r)) < +\infty$

"$\Rightarrow$" $\forall$ compact K $\exists B(0,n)$, s.t $K \subset B(0,n)$

$\mu(K) = \int_K f(x) dx \leq \int_{B(0,n)} f(x) dx < +\infty$

outer regular:

设A为任意 Borel 集

① A有界 $\exists R$ s.t $A \subset B(0,R)$

$\bar{\mu}(\overline{B(0,R)}) < +\infty$

$\exists O$. s.t $m(O) - m(A) = m(O \setminus A) < \delta$. $O^* = O \cap B(0,R)$

$m(O^* \setminus A) < \delta$.

绝对连续 $\int_{O^* \setminus A} f dx < \varepsilon$. $A \subset O^* \subset \overline{B(0,R)}$ $\int_A f dx < +\infty$

$$\int_{O^*} f dx < \int_A f dx + \varepsilon$$

$\therefore \forall \varepsilon > 0$ $\exists O' \supset A$. $\int_{O'} f dx < \int_A f + \varepsilon$

$\mu(O') < \mu(A) + \varepsilon$. 对有界集 ✓

② A无界 不失一般 设 $\mu(A) < +\infty$

$A_i = A \cap (B(0,i) \setminus B(0,i-1))$ $i \geq 2$ $A_1 = A \cap B(0,1)$

$\int_{O_n} f < \int_{A_n} + \frac{\varepsilon}{2^n}$ $O = \bigcup_{n=1}^{\infty} O_n \supset \bigcup_{n=1}^{\infty} A_n = A$.

$\mu(O) \leq \int_A f + \varepsilon = \mu(A) + \varepsilon$. #

补充：

1. $f^* = \sup_{x\in B} \frac{1}{m(B)}\int_B |f(y)|dy$

$m(f^*>\lambda) \le \frac{A}{\lambda}\int_{\{f^*>\lambda\}}|f(x)|$ $\quad E=\{x \mid f^*(x)>\lambda\}$. $\int_E |f(x)|dx$

$=\frac{A}{\lambda}\|f\chi_{\{f^*>\lambda\}}\|_{L^1}$.

proof: $f_1=f\cdot\chi_{\{f^*>\lambda\}}$ $\quad f_2 = f\cdot\chi_{\{f^*\le\lambda\}}$ $\quad f=f_1+f_2$.

只需证明 $\{f^*>\lambda\}\subset\{f_1>\lambda\}\le\frac{A}{\lambda}\|f_1\|_{L^1}$

若 $x_0\in\{f^*>\lambda\}$ $\quad f^*(x_0)>\lambda$. $\quad \exists B$ s.t. $x_0\in B$

$\frac{1}{m(B)}\int_B|f(y)|dy>\lambda$. $\quad \forall x\in B$. $f^*(x)>\lambda$. $\quad x\in\{f^*>\lambda\}$ $\quad B\subset\{f^*>\lambda\}$

$\because B\subset\{f^*>\lambda\}$ $\quad \therefore f_1(y)=f(y)$ 当 $y\in B$ 时

$f_1^*(x_0)\ge\frac{1}{m(B)}\int_B|f_1(y)|dy = \frac{1}{m(B)}\int_B|f(y)|dy>\lambda$.

$\therefore x_0\in\{f_1^*>\lambda\}$

2. $0<\delta<1$ $\quad m(f^*>\lambda)\le\frac{A}{(1-\delta)\lambda}\int_{\{|f|>\delta\lambda\}}|f(x)|dx$

proof: $f_1=f\chi_{\{|f|>\delta\lambda\}}$ $\quad f_2=f\cdot\chi_{\{|f|\le\delta\lambda\}}$ $\quad f=f_1+f_2$.

$f^*\le f_1^*+f_2^*\le f_1^*+\delta\lambda$

$\{f^*>\lambda\}\subset\{f_1^*+\delta\lambda>\lambda\}=\{f_1^*>(1-\delta)\lambda\}$

$m(f^*>\lambda)\le m(f_1^*>(1-\delta)\lambda)\le\frac{A}{(1-\delta)\lambda}\|f_1\|_{L^1}=\frac{A}{(1-\delta)\lambda}\int_{\{|f|>\delta\lambda\}}|f(x)|dx$

$\int_{\{|f_1|>(1-\delta)\lambda\}}|f_1|dx\le\|f_1\|_{L^1}$

3. $m(E)<+\infty$. $\quad 0<\delta<1$

$\int_E f^*(x)dx\le\frac{1}{\delta}m(E)+\frac{A}{1-\delta}\int_{\{|f|\ge1\}}|f(x)|\ln|f(x)|dx$.

proof: $\because \int_{\mathbb{R}^d}|f(x)|dx=\int_0^{+\infty}m(\{|f|>\lambda\})d\lambda$

$\therefore \int_E f^*(x)dx=\int_0^{+\infty}m(E\cap\{f^*>\lambda\})d\lambda=\int_0^{\frac{1}{\delta}}+\int_{\frac{1}{\delta}}^{+\infty}\le\frac{1}{\delta}m(E)$

$+\int_{\frac{1}{\delta}}m(f^*>\lambda)d\lambda \le\frac{1}{\delta}m(E)+\int_{\frac{1}{\delta}}^{+\infty}\left(\frac{A}{(1-\delta)\lambda}\int_{\{|f|>\delta\lambda\}}|f|dx\right)d\lambda$

$$=\frac{1}{\delta}m(E)+\frac{A}{1-\delta}\int_{\mathbb{R}^d}\int_{\frac{1}{\delta}}^{+\infty}\frac{1}{\lambda}|f(x)|\,\chi_{\{|f(x)|>\delta\lambda\}}(x,\lambda)\,d\lambda\,dx$$

$$=\frac{1}{\delta}m(E)+\frac{A}{1-\delta}\int_{|f|\geq 1}\int_{\frac{1}{\delta}}^{+\infty}\frac{1}{\lambda}|f(x)|\,\chi_{\{|f(x)|>\delta\lambda\}}(x,\lambda)\,d\lambda\,dx$$ 若$|f(x)|<1$ $\lambda<\frac{1}{\delta}$

$$=\frac{1}{\delta}m(E)+\frac{A}{1-\delta}\int_{\{|f|\geq 1\}}\int_{\frac{1}{\delta}}^{\frac{|f(x)|}{\delta}}\frac{1}{\lambda}|f(x)|\,d\lambda\,)dx=\frac{1}{\delta}m(E)+\frac{A}{1-\delta}\int_{\{|f|\geq 1\}}|f(x)|\ln|f(x)|\,dx$$ 特征函数加

f^* 可以在有限区间上可积

PSet 14 - 1

1. $F: \mathbb{R} \to \mathbb{R}$. Pf:

① $\|F + G\|_{TV} \le \|F\|_{TV} + \|G\|_{TV}$

② $\|cF\|_{TV} = |c|\|F\|_{TV}$

③ $\|F\|_{TV} = 0 \iff F \equiv C$

④ $a < b < c \Rightarrow \|F\|_{TV([a,c])} = \|F\|_{TV([a,b])} + \|F\|_{TV([b,c])}$

Pf: ① $\forall\ x_0 < x_1 < \cdots < x_n < x_{n+1}$

$$\sum_{k=0}^{n} |(F+G)(x_{k+1}) - (F+G)(x_k)|$$

$$\le \sum |F(x_{k+1}) - F(x_k)| + \sum |G(x_{k+1}) - G(x_k)|$$

$$\le \|F\|_{TV} + \|G\|_{TV}$$

$\therefore\ \|F+G\|_{TV} \le \|F\|_{TV} + \|G\|_{TV}$

② $\|cF\|_{TV} = \sup \sum |cF(x_{k+1}) - cF(x_k)|$

$$= |c| \sup \sum |F(x_{k+1}) - F(x_k)|$$

$$= |c| \|F\|_{TV}$$

③ "$\Leftarrow$" 显然

"$\Rightarrow$" 若不然, $F(x) \ne F(y)$

则 $\|F\|_{TV} \ge |F(x) - F(y)| > 0$ 矛盾

④ $a = x_0 < \cdots < x_{n+1} = b = y_0 < \cdots < y_{m+1} = c$

则 $\|F\|_{TV([a,c])} \ge \sum |F(x_{k+1}) - F(x_k)| + \sum |F(y_{l+1}) - F(y_l)|$

取 sup $\Rightarrow \|F\|_{TV([a,c])} \ge \|F\|_{TV([a,b])} + \|F\|_{TV([b,c])}$

$\forall \varepsilon > 0$, $\exists\ a = x_0 < \cdots < x_{n+1} = c$ ~~设~~ $\|F\|_{TV([a,c])} \le \sum |F(x_{k+1}) - F(x_k)| + \varepsilon$

设 $x_t \le b < x_{t+1}$,

则 $\|F\|_{TV([a,c])} - \varepsilon \le \sum |F(x_{k+1}) - F(x_k)| \le \sum_{k=0}^{t-1} |F(x_{k+1}) - F(x_k)| + |F(b) - F(x_t)| + |F(x_{t+1}) - F(\ldots$

$\le \|F\|_{TV([a,b])} + \|F\|\ldots$

2. $D^+F = \limsup\limits_{h\to 0^+} \dfrac{F(x+h)-F(x)}{h}$

Pf: ① D^+F 可测

② $a<b$, $\lambda>0$

$m(\{D^+F>\lambda\}) \le \frac{3}{\lambda}(F(b)-F(a))$

Pf: ① 设 $D^+F = \limsup\limits_{h\to 0^+} \dfrac{F(x+h)-F(x)}{h}$, $G(x) = \limsup\limits_{h\to 0^+,\ h\in\mathbb{Q}} \dfrac{F(x+h)-F(x)}{h}$

$\forall \varepsilon>0$, 取 $h_1, h_2 \in \mathbb{Q}$ $h_1<h<h_2$ s.t. $\frac{h_1}{h}>1-\varepsilon$, $\frac{h_2}{h}<1+\varepsilon$

而 $(1-\varepsilon)\dfrac{F(x+h_1)-F(x)}{h_1} < \dfrac{F(x+h)-F(x)}{h} \le \dfrac{F(x+h_2)-F(x)}{h_2}\cdot\dfrac{h_2}{h} < (1+\varepsilon)\dfrac{F(x+h_2)-F(x)}{h_2}$

~~令 $h\to 0^+$，得~~ 取 $\limsup\limits_{h\to 0^+}$，得

$(1-\varepsilon)G(x) \le D^+F(x) \le (1+\varepsilon)G(x)$

$\therefore D^+F(x) = G(x)$, $\Rightarrow D^+F$ 可测

② F 单调 故 $D^+F = F'$ a.e.

$\therefore m(\{D^+F>\lambda\}) = m(\{F'>\lambda\}) \le \frac{1}{\lambda}\int_a^b F'\,dx \le \cancel{F(b)}\ \frac{1}{\lambda}(F(b)-F(a))$

$\le \frac{3}{\lambda}(F(b)-F(a))$

3. (1) $F, G \sim AC[a,b]$

Pf: $F+G$, FG AC

(2) $F: \mathbb{R}\to\mathbb{R} \sim AC$, $A\subset\mathbb{R}$ null

Pf: $F(A)\subset\mathbb{R}$ null

Pf: (1) $\cancel{F(x)}$ $(F+G)(y)-(F+G)(x) = F(y)-F(x)+G(y)-G(x)$

$= \int_x^y F' + \int_x^y G' = \int_x^y F'+G'$

$\therefore F+G$ AC

$F, G \in AC$, $FG \in C([a,b])$ 设 $F\le M$, $G\le N$

则 $|FG(y)-FG(x)| \le M|G(y)-G(x)| + N|F(y)-F(x)|$

$\therefore FG \sim AC$

(2) $\forall \varepsilon>0,\ \exists \delta>0,\ \forall\ \sum(b_i-a_i)<\delta,\ A\subset \bigcup(a_i,b_i)$

$F(A)\subset \bigcup F((a_i,b_i))$

$\therefore m(F(A))\le \sum m(F[a_i,b_i])$

$\le \sum|F(b_i')-F(a_i')|<\varepsilon$

$b_i'=\arg\max F(x)$

$a_i'=\arg\min F(x)$

$(a_i',b_i')\subset(a_i,b_i)$

$\therefore m(F(A))=0$

4. $\{F_j\}$ 非负↗，$F(x):=\sum F_j(x)<\infty$

Pf: $F'(x)=\sum_{j=1}^{\infty}F_j'(x)$ a.e.

~~Pf: 不妨设 F_j 右连续，则 F 右连续~~

~~$F=\sum F_j<\infty \Rightarrow \mu_F:=\sum\mu_{F_j}$ 为正测度~~

~~$\therefore \frac{d(\mu_F)_r}{dm}=\sum_{j=1}^{\infty}\frac{d(\mu_j)_r}{dm}$~~

~~$(\mu_F)_r(A)$~~ ~~$\frac{d(\mu_F)_r}{dm}=\sum\frac{d(\mu_j)_r}{dm}$ (MCT)~~

~~$\therefore F'(x)=\sum_j F_j'(x)$~~

~~Pset 14-2~~ Pf: 不妨设 F_j 右连续，则 F 右连续

~~F.~~

则 $\mu_F=\sum_{j=1}^{\infty}\mu_{F_j}$

设 $(\ )_{ac}$，$(\ )_s$ 为关于 m 的 AC，singular 部分

则 $\mu_F=\sum_{j=1}^{\infty}\mu_{F_j}=\sum_{j=1}^{\infty}(\mu_{F_j})_{ac}+\sum_{j=1}^{\infty}(\mu_{F_j})_s$，且 $\begin{cases}\sum_{j=1}^{\infty}(\mu_{F_j})_{ac}\ll m\\ \sum_{j=1}^{\infty}(\mu_{F_j})_s\perp m\end{cases}$ (由 Pset 13-1 T2)

由分解唯一性

$(\mu_F)_{ac}=\sum(\mu_{F_j})_{ac},\quad (\mu_F)_s=\sum(\mu_{F_j})_s$

而 $(\mu_F)_{ac}(A)=\sum(\mu_{F_j})_{ac}(A)=\sum\int_A\frac{d(\mu_{F_j})_{ac}}{dm}dm\overset{MCT}{=}\int_A\sum\frac{d\mu_{F_j}}{dm}dm=\int_A\sum F_j'\,dm$

$\therefore F'=\frac{d(\mu_F)_{ac}}{dm}=\sum F_j'$

PSet 14 -2

1. $\{u_\alpha\}_{\alpha\in I}$ lip on X, $lip(u_\alpha)\le L$

(1) $U(x)=\sup_\alpha u_\alpha(x)<\infty$ at x_0. Pf: $Lip(U)\le L$

(2) $u(x)=\inf_\alpha u_\alpha(x)$ . . . . $Lip(u)\le L$

Pf: (1) $\forall y\in X$

$|u_\alpha(x_0)-u_\alpha(y)|\le L\,d(x_0,y)$

$\therefore\ U(y)<\infty,\ \forall y$

而 $U(x)=\sup_\alpha u_\alpha(x)\le\sup(u_\alpha(x)-u_\alpha(y))+\sup u_\alpha(y)$

$\le L\,d(x,y)+U(y)$

$\therefore\ U(x)-U(y)\le L\,d(x,y)$

反面类似

(2) 令 $U=-u$ 即得

2. X ~ metrix space, $A\subset X$

$\forall$ lip $f: A\to\mathbb{R}$, $lip(f)=L$. $\tilde f: X\to\mathbb{R}$. $\tilde f=\inf_{y\in A}(f(y)+L\,d(x,y))$

Pf: $\tilde f$ lip, $Lip(\tilde f)\le L$

Pf: 令 $g_y(x)=f(y)+L\,d(x,y)$

则 $|g_y(x)-g_y(x')|=L\,|d(x,y)-d(x',y)|\le L\,d(x,x')$

$\therefore\ Lip(g_y)\le L$

而 $|f(y)-f(x_0)|\le L\,d(y,x_0)$

$\Rightarrow\ f(x_0)\le f(y)+L\,d(x_0,y)$

$\cancel{-\infty<f(x_0)\le}\ \tilde f(x_0)\ge f(x_0)>-\infty$

$\therefore$ 由1，可得：

3. f 凸, Prove:

① f diff

② $f' \nearrow$

Pf: ① $\forall\ a < x < b$, $x = (1-t)x + ty$

$$f(x) \le (1-t)f(a) + tf(b)$$

$$\therefore \frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a}$$

类似, $\dfrac{f(x)-f(b)}{x-b} \ge \dfrac{f(b)-f(a)}{b-a}$

进一步, 我们有 $x < x' < y < y'$

$$\boxed{\frac{f(x')-f(x)}{x'-x} \le \frac{f(y)-f(x')}{y-x'} \le \frac{f(y')-f(y)}{y'-y}}$$

$\forall$ 区间 $[a,b]$, 取 $c<a$, $b<d$

$\forall\ x,y \in [a,b]$, $\dfrac{f(a)-f(c)}{a-c} \le \dfrac{f(x)-f(y)}{x-y} \le \dfrac{f(d)-f(b)}{d-b}$

令 $\therefore\ lip(f|_{[a,b]}) \le \max\left\{\left|\dfrac{f(a)-f(c)}{a-c}\right|, \left|\dfrac{f(d)-f(b)}{d-b}\right|\right\}$

$\therefore$ f 在 $[a,b]$ 上 lip $\therefore$ f 局部 lip, ~~处处~~几乎处处可微

② $\forall\ x < y$, $h < |x-y|$

$$\frac{f(x+h)-f(x)}{h} \le \frac{f(y+h)-f(y)}{h}$$

令 $h \to 0$ $\quad f'(x) \le f'(y)$

4. (1) $\int h\varphi = 0 \quad \forall \varphi \in C_0^\infty \Rightarrow h = 0$ a.e.

(2) LDT ~

Pf. $\forall x \in \mathbb{R}^d$, Leb point of ~~考~~ h

考虑 令 $F(x) = \int_0^x h\, dm$

则 $(F\varphi)' = F\varphi' + h\varphi \quad \forall [a,b], \ \operatorname{spt}\varphi \subset (a,b)$

~~$F\varphi(y) - F(x)$~~

$\therefore \int_a^b F\varphi' = 0 \quad \forall \varphi \subset\subset (a,b)$

x. Leb点, 有界

Pf: $\forall\ \overline{B(x,r)}$, $U_1 \supset U_2 \supset \cdots \supset \overline{B(x,r)}$, $\bigcap U_i = \overline{B(x,r)}$

$\operatorname{spt} \varphi_j \subset U_j$, $\varphi_j = \dfrac{1}{\overline{B(x,r)}}$ on $\overline{B(x,r)}$ $0 \le \varphi_j \le \dfrac{1}{\overline{B(x,r)}}$

则 $\int h\varphi_j = 0 \qquad |h\varphi_j| \le |h|\chi_{U_1}$ 由DCT

$j \to \infty \Rightarrow \dfrac{1}{m(B(x,r))} \int_{\overline{B(x,r)}} h = 0$

$\therefore h(x) = 0$ a.e.