Real Analysis Problem Set 1, Part 2

03/01/2018

$\boxed{\text{Due 03/05, before class}}$

1. Prove: (1) If $E_1, E_2 \in \mathcal{E}_0$, so is $E_1 \cup E_2$, $E_1 \cap E_2$ and $E_1 \setminus E_2$.

   (2) If $A_1, A_2 \in \mathcal{J}$, so is $A_1 \cup A_2$, $A_1 \cap A_2$ and $A_1 \setminus A_2$.

2. Read the bottom of page 2 of the course (lecture 2) notes. Complete the proof the following theorem.

   Thm: If $\tilde{m}: \mathcal{E}_0 \to [0, +\infty)$ is finite-additive, translation-invariant and $\tilde{m}([0,1]^d) = 1$, then $\tilde{m} = m$.

3. Prove: (1) If $E_1 \subset \mathbb{R}^k$, $E_2 \subset \mathbb{R}^l$ are elementary sets, so is $E_1 \times E_2$, and
$$m(E_1 \times E_2) = m(E_1) \times m(E_2)$$
   (2) If $A_1 \subset \mathbb{R}^k$, $A_2 \subset \mathbb{R}^l$ are Jordan measurable, so is $A_1 \times A_2$, and
$$J(A_1 \times A_2) = J(A_1) \times J(A_2)$$

4. Prove the discretization formula:

   (1) If B is a box in $\mathbb{R}^d$, then $m(B) = \lim_{N \to \infty} \frac{1}{N^d} \#(B \cap \frac{1}{N}\mathbb{Z}^d)$

   (2) If E is an elementary set in $\mathbb{R}^d$, then $m(E) = \lim_{N \to \infty} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbb{Z}^d)$

   (3) If $A \subset \mathbb{R}^d$ is Jordan measurable, then $m(A) = \lim_{N \to \infty} \frac{1}{N^d} \#(A \cap \frac{1}{N}\mathbb{Z}^d)$.

   (4) Find a set $C \subset [0,1]$ such that the limit
$$\lim_{N \to \infty} \frac{1}{N} \#(C \cap \frac{1}{N}\mathbb{Z})$$
   does not exist.