05/08/2018

Real Analysis PSet 10, Part 1

Due 05/14, before class

1. Read Course Notes, page 9 – page 11.

2. (1) Prove the Dirac delta measure δ_{x_0} (see Lecture 15, page 5) on a locally compact σ-compact metric space X is a Radon measure.

   (2) Suppose μ_1, μ_2 are two Radon measures on X. Prove: $\mu_1 + \mu_2$ is a Radon measure.

   (3) Let $\mu: \mathcal{B} \to [0, +\infty]$ be a Borel measure on $\mathbb{R}^d$, s.t. $\mu(B(0,r)) < +\infty$, $\forall r$.
   Prove: μ is a Radon measure.

3. Let $X = \mathbb{R}$, with $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$.

   (1) Let $\mathcal{F} = \{A \subset \mathbb{R} : A \text{ is countable, or } A^c \text{ is countable}\}$. Prove: $\mathcal{F}$ is a σ-algebra.

   (2). What are compact sets in (X, d)? What is $C_c(X)$?

   (3). Let $\mu: \mathcal{F} \to [0, +\infty]$ be $\mu(A) = \begin{cases} 1, & \text{if } A^c \text{ is countable} \\ 0, & \text{if } A \text{ is countable} \end{cases}$
   Prove: μ is a measure on $\mathcal{F}$.

   (4) Prove: $\forall f \in C_c(X)$, $I_\mu(f) := \int_X f \, d\mu = 0$. [In other words, $I_\mu = I_0$]

4. Let (X, d) be a compact metric space, and $\ell: C(X) \to \mathbb{R}$ is a linear functional. Moreover, assume ℓ is <u>bounded</u>, in the sense that $\exists C > 0$ s.t. $\forall f \in C(X)$,
   $$\ell(f) \leq C \cdot \sup_X |f|.$$

   (1) For any $g \in C(X)$ s.t. $g \geq 0$, define $\ell^+(g) = \sup_{0 \leq \varphi \leq g} \ell(\varphi)$.
   <u>Prove</u>: $\ell^+(g) \in \mathbb{R}$, and
   - $\ell^+(g_1 + g_2) = \ell^+(g_1) + \ell^+(g_2)$, $\forall g_1, g_2 \in C(X)$, $g_1, g_2 \geq 0$.
   - $\ell^+(cg) = c\,\ell^+(g)$, $\forall g \in C(X)$, $g \geq 0$, $c \geq 0$.

   (2) For any $g \in C(X)$. Take constant M s.t. $M \geq 0$ and $g + M \geq 0$.
   Define $\ell^+(g) = \ell^+(g+M) - \ell^+(M)$. <u>Prove</u>: $\ell^+: C(X) \to \mathbb{R}$ is a well-defined positive linear functional.

   (3). Prove: $\ell^- := \ell^+ - \ell$ is a positive linear functional on $C(X)$.

   (4) Conclude that there exists Borel measures μ_1, μ_2 on X s.t.
   $$\ell(f) = I_{\mu_1}(f) - I_{\mu_2}(f), \quad \forall f \in C(X).$$