05/10/2018

Real Analysis Problem Set 10, Part 2

Due 05/14, before class

1. (1) Young's inequality is an equality iff $a^p = b^q$.

   (2) When will the Hölder inequality be an equality?

   (3) ------ Minkowski ------------? [Note: what if $p=1$?]

2. Suppose $f, g \in L^p(X)$, $p>1$. Let
$$F(t) = \int_X |f + tg|^p \, d\mu.$$
<u>Prove</u>: $\frac{dF}{dt}(0) = p \int_X |f|^{p-2} f g \, d\mu$.

   [Note: $|f|^{p-2} f$ is well-defined even if $f=0$, $\forall 1<p<2$. and is in L^q.]

3. Suppose $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$ (so $q<0$). [We assume: $g \neq 0$ a.e.]

   (1) <u>Prove</u>: $\forall f \in L^p(X)$, $g \in L^q(X)$, $\|f\|_{L^p} \cdot \|g\|_{L^q} \le \|fg\|_{L^1}$.

   (2) <u>Prove</u>: $\forall f,g \in L^p(X)$, $\|f+g\|_{L^p}^p \le \|f\|_{L^p}^p + \|g\|_{L^p}^p$

   (3) <u>Prove</u>: $\forall f,g \in L^p(X)$, $\|f\|_{L^p} + \|g\|_{L^p} \le \| |f|+|g| \|_{L^p} \le 2^{\frac{1}{p}-1}\left(\|f\|_{L^p} + \|g\|_{L^p}\right)$.

4. Let μ be a finite measure on X, i.e. $\mu(X) < +\infty$.

   (1) <u>Prove</u>: $\lim_{p \to p_0^-} \|f\|_{L^p} = \|f\|_{L^{p_0}}$, $\forall 0<p_0<+\infty$

   (2) <u>Prove</u>: If $0<p_1<p_2<\infty$, then $\|f\|_{L^{p_1}} \le (\mu(X))^{\frac{1}{p_1}-\frac{1}{p_2}} \|f\|_{L^{p_2}}$.

   (3) Prove: $L^{p_2}(X) \subset L^{p_1}(X)$. Do we have the reverse?

   (4) Suppose f, g are non-negative, and $f \cdot g \ge 1$. Prove: $\int_X f \, d\mu \cdot \int_X g \, d\mu \ge (\mu(X))^2$