Real Analysis Problem Set 11, Part 1

05/14/2018

$\boxed{\text{Due 05/21, before class}}$

1. Suppose $0 < p < q < r \le +\infty$. Prove:

   (1) $L^q \subset L^p + L^r$ (i.e. $\forall f \in L^q$, $\exists g \in L^p$, $h \in L^r$ s.t. $f = g + h$.)

   (2) $L^p \cap L^r \subset L^q$. Moreover, $\forall f \in L^p \cap L^r$,
   $$\|f\|_{L^q} \le \|f\|_{L^p}^{\lambda} \|f\|_{L^r}^{1-\lambda}, \quad \left(\text{where } \lambda \in (0,1) \text{ is determined by } \frac{1}{q} = \frac{\lambda}{p} + \frac{(1-\lambda)}{r}\right).$$

2. (1) Suppose $f \in L^\infty \cap L^{p_0}$ for some $0 < p_0 < \infty$. Prove: $\lim_{p \to \infty} \|f\|_{L^p} = \|f\|_{L^\infty}$.

   (2) Suppose $f \notin L^\infty$. Prove: $\lim_{p \to \infty} \|f\|_{L^p} = \infty$.

3. Suppose $1 \le p < \infty$. Let $f_n \in L^p$, $f \in L^p$, and $f_n \to f$ a.e.

   Prove: $\|f_n - f\|_{L^p} \to 0$ if and only if $\|f_n\|_{L^p} \to \|f\|_{L^p}$.

4. Suppose $\mu(X) < +\infty$, $1 \le p_1 < p_2 < +\infty$.

   (1). Prove: $\|\cdot\|_{L^{p_1}}$ is a norm on $L^{p_2}(X)$.

   (2). Is the space $\left(L^{p_2}(X), \|\cdot\|_{L^{p_1}}\right)$ a Banach space?