Real Analysis Problem Set 11, Part 2

05/17/2018

Due 05/21, before class

1. Let V be any complex vector space.
   (1) Define the conception of inner product $\langle \cdot, \cdot \rangle$ on V.
   (2) Prove the Cauchy-Schwarz inequality $|\langle v, w\rangle| \le \|v\| \cdot \|w\|$.
   (3) Prove: $\|v\| := \sqrt{\langle v, v\rangle}$ is a norm on V.

2. (1) For any $p \neq 2$, find $f, g \in L^p([0,1])$ s.t. $\|f+g\|_{L^p}^2 + \|f-g\|_{L^p}^2 \neq 2(\|f\|_{L^p}^2 + \|g\|_{L^p}^2)$.
   (2) Suppose V is a normed vector space, s.t. $\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$.
   Prove: One can define an inner product $\langle \cdot, \cdot \rangle$ on V s.t. $\|v\|^2 = \langle v, v\rangle$.

3 (1) Let $V = C([0,1])$, endowed with the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.
   Let $K \subset V$ be the subspace $K = \{f \in C([0,1]) : \operatorname{supp}(f) \subset [0, \tfrac{1}{2}]\}$.
   Prove: • K is closed (i.e. $x_n \in K$, $x_n \to x \Rightarrow x \in K$) and convex.
   • The "existence of minimizer" property fails in this example
   (2). Prove: The "existence of minimizer" property fails for ℓ^∞.

4. (1) Read 周民强, 第三版, Page 266-271.
   (2) Start with Thm 6.15 on Page 270. Prove: $\|f\|_{L^2}^2 = \sum_{k=1}^{\infty} |\langle f, \varphi_k\rangle|^2$.

[Parseval's identity]