Real Analysis Problem Set 12, Part 1

05/24/2018

Due 05/28, before class

1. Let V be a vector space, and $\|\cdot\|$ a norm on V.
For any continuous linear functional $\ell: V \to \mathbb{R}$, we define
$$\|\ell\|_{V^*} = \sup\{|\ell(v)| : \|v\| = 1\}.$$
Let V^* be the set of all continuous linear functionals on V.
Prove: V^* is a vector space, and $\|\cdot\|_{V^*}$ is a norm on V^*.

2. Suppose $1 < p \leq \infty$, $1 \leq q < \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.
For any $g \in L^q$, we define the norm of $L_g: L^p(X) \to \mathbb{R}$ as in problem 1, i.e.
$$\|L_g\|_{(L^p(X))^*} = \sup\left\{\int_X f g \, d\mu : \|f\|_{L^p(X)} = 1\right\}.$$
Prove: $\|L_g\|_{(L^p(X))^*} = \|g\|_{L^q(X)}$

3. We say $v_n \in V$ <u>converges weakly</u> to $v \in V$ if for any continuous linear functional ℓ, one has $\ell(v_n) \to \ell(v)$ as $n \to \infty$.
(1) Let $f_n(x) = \cos(2\pi n x)$. Prove: $f_n(x)$ converges weakly to 0 in $L^2([0,1])$.
(2) Does $f_n \to 0$ in measure? a.e.?

4. Let $f_n(x) = n \cdot \chi_{(0,\frac{1}{n})}$. Prove: $f_n \to 0$ in measure, a.e., but $\forall p \geq 1$, f_n does not converges weakly to 0 in $L^p(X)$.