05/31/2018

Real Analysis Problem Set 13, Part 2

Due 06/04, before class

1. Prove the Lebesgue differentiation theorem. (Page 4, line 3-6, of the notes).

2. Let $f \in L^1(\mathbb{R}^d)$. Consider the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t)|\,dt.$$

Prove: (1) Mf is measurable

(2) $Mf(x) < +\infty$ for a.e. $x \in \mathbb{R}^d$

(3) $Mf(x) \geq |f(x)|$, a.e. $x \in \mathbb{R}^d$.

(4). Suppose f is NOT identically zero. Prove: Mf is not integrable.

3. Prove the following Vitali covering lemma:

Let $A \subset \mathbb{R}^d$ be a measurable set with $m(A) < \infty$.

Let $\mathcal{U} = \{B(x_\alpha, r_\alpha)\}$ be a Vitali covering of A, i.e. $\forall x \in A$, $\forall r > 0$, $\exists B \in \mathcal{U}$, s.t. $x \in B$ and the radius of $B < r$.

Prove: $\forall \varepsilon > 0$ One can find finitely many balls $B_1, \cdots, B_N$ in $\mathcal{U}$ that are pairwise disjoint, s.t. $\sum_{i=1}^{N} m(B_i) \geq m(A) - \varepsilon$.

4. Let $f: \mathbb{R}^d \to [0, +\infty]$ be a measurable function. Consider the Borel measure $d\mu = f\,dx$.

Prove: $f \in L^1_{loc}(\mathbb{R}^d) \Leftrightarrow \mu$ is locally finite and outer regular.