06/04/2018

Real Analysis Problem Set 14, Part 1

$\boxed{\text{Due 06/11, before class}}$

1. Prove the following properties of the total variation of a function $F: \mathbb{R} \to \mathbb{R}$:

   ① $\|F+G\|_{TV} \le \|F\|_{TV} + \|G\|_{TV}$

   ② $\|cF\|_{TV} = |c| \cdot \|F\|_{TV}$

   ③ $\|F\|_{TV} = 0 \Leftrightarrow F$ is a constant

   ④ If $a<b<c$, then $\|F\|_{TV([a,c])} = \|F\|_{TV([a,b])} + \|F\|_{TV([b,c])}$.

2. Consider the four Dini derivatives of a function F. (For definition, read page 204-205 of your book).
   Suppose $F: \mathbb{R} \to \mathbb{R}$ is monotone increasing. Prove

   (1) Each of the four Dini derivatives D^+F, D_+F, D^-F, D_-F is measurable.

   (2) For any $a<b$ and any $\alpha>0$, one has Hardy-Littlewood type inequality

   $$m\left(\{x \in [a,b] : D^+F(x) > \alpha\}\right) \le \frac{3}{\alpha}\left(F(b) - F(a)\right).$$

   [and the same holds if we replace D^+F by D_+F or D^-F or D_-F]

3. (1) Let F, G be AC functions on $[a,b]$.
   Prove: Both $F+G$ and $F \cdot G$ are AC functions.

   (2) Let $F: \mathbb{R} \to \mathbb{R}$ be a AC function, and $A \subset \mathbb{R}$ is a null set.
   Prove: $F(A) \subset \mathbb{R}$ is also a null set.

4. Let $\{F_j\}$ be a sequence of non-negative increasing functions on $[a,b]$ such that

   $$F(x) := \sum_{j=1}^{\infty} F_j(x) < +\infty, \quad \forall x \in [a,b]$$

   Prove: $F'(x) = \sum_{j=1}^{\infty} F_j'(x)$, a.e. $x \in [a,b]$.