# Real Analysis Problem Set 14, Part 2

06/07/2018

Due 06/11, before class

1. Let $\{u_\alpha\}_{\alpha \in I}$ be a family of Lipschitz functions on a metric space $X$.
   Assume $Lip(u_\alpha) \le L$, $\forall \alpha$.
   (1) Suppose $U(x) := \sup_\alpha u_\alpha(x)$ is finite at one point. Prove. $Lip(U) \le L$.
   (2) ---. $u(x) := \inf_\alpha u_\alpha(x)$ ---------- ---- $Lip(u) \le L$.

2. Let $X$ be a metric space, and $A \subset X$ a subset.
   Given any Lipschitz function $f: A \to \mathbb{R}$ with $Lip(f) = L$, define
   $$\tilde{f}: X \to \mathbb{R}, \quad \tilde{f}(x) = \inf_{y \in A} \left( f(y) + L \cdot d(x,y) \right)$$
   Prove. $\tilde{f}$ is a Lipschitz function, and $Lip(\tilde{f}) \le L$.

3. Let $f: \mathbb{R} \to \mathbb{R}$ be a convex function, i.e.
   $$f((1-t)x + ty) \le (1-t)f(x) + tf(y), \quad \forall x<y, \ 0<t<1.$$
   Prove: ① $f$ is a.e. differentiable
   ② $f'$ equal a.e. to a monotone increasing function. ($\rightarrow f''$ a.e.)

4. (1) Read the lemma on the top of page 6. ($\int h\varphi = 0$, $\forall \varphi \in C_0^\infty \Rightarrow h = 0$ a.e.)
   (2) Give another proof, using the Lebesgue differentiation theorem