Real Analysis Problem Set 2, Part 1

03/05/2018

Due 03/12, before class

1. Let $A \subset \mathbb{R}^d$ be any bounded set. Prove $J_*(A) \le m^*(A) \le J^*(A)$.
   As a consequence, prove that if A is Jordan measurable, then $m^*(A) = J(A)$.

2. Prove: every open set $A \subset \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

3. Let $E, F \subset \mathbb{R}^d$. Suppose $d(E,F) := \inf\{|x-y| \mid x \in E,\ y \in F\} > 0$.
   Prove: $m^*(E \cup F) = m^*(E) + m^*(F)$.

4. For any bounded set $A \subset \mathbb{R}^d$, define the <u>Lebesgue inner measure</u> of A to be

$$m_*(A) := m(E) - m^*(E \setminus A),$$

   where E is any elementary set s.t. $A \subset E$.

   (1) Prove: m_* is well-defined, i.e. if E' is another elementary set s.t. $A \subset E'$, then $m(E') - m^*(E' \setminus A) = m(E) - m^*(E \setminus A)$.

   (2) Prove: $m_*(A) = \sup\limits_{\substack{K \subset A \\ K \text{ is compact}}} m_*(K)$.

   (3) Prove super-additivity: If $A_1 \cap A_2 = \phi$, then $m_*(A_1 \cup A_2) \ge m_*(A_1) + m_*(A_2)$.