Real Analysis Problem Set 2, Part 2

03/08/2018

$\boxed{\text{Due 03/12, before class}}$

1. Prove: Any Jordan measurable set is Lebesgue measurable.

2. The symmetric difference of two sets is $A_1 \Delta A_2 := (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$.

   Prove: $A \subset \mathbb{R}^d$ is Lebesgue measurable

   $\Leftrightarrow \forall \varepsilon > 0$, $\exists$ open set U s.t. $m^*(U \Delta A) < \varepsilon$ ("nearly open")

   $\Leftrightarrow \forall \varepsilon > 0$, $\exists$ closed set F s.t. $m^*(F \Delta A) < \varepsilon$ ("nearly closed")

   $\Leftrightarrow \forall \varepsilon > 0$, $\exists$ Lebesgue measurable set C s.t. $m^*(C \Delta A) < \varepsilon$. ("nearly measurable")

3. Suppose $A \subset \mathbb{R}^d$ is Lebesgue measurable. Prove: $m(A) = \sup\limits_{\substack{A \supset K \\ K \text{ compact}}} m(K)$.

   "Lebesgue inner measure" see Part 1 of PSet 2.

4. Let $\mathbb{R}^d \supset A_1 \supset A_2 \supset \cdots$ be a sequence of "decreasing" Lebesgue measurable sets. Suppose one of $m(A_n)$ is finite.

   Prove $$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} m(A_n).$$

   Show by counterexample that why we need to assume one of $m(A_n)$ is finite.