03/15/2018

Real Analysis Problem Set 3, Part 2

$\boxed{\text{Due 03/19, before class}}$

we always assume $A \in \mathcal{L}$ in this PSet.

1. Prove: A real-valued function $f: A \subset \mathbb{R}^d \to \mathbb{R}$ is measurable
   $\Leftrightarrow$ For any open set $U \subset \mathbb{R}$, $f^{-1}(U) \in \mathcal{L}$.
   $\Leftrightarrow$ For any closed set $F \subset \mathbb{R}$, $f^{-1}(F) \in \mathcal{L}$.

2. Suppose f, g are measurable functions on A. Prove: the sets
   $$\{x \in A : f(x) > g(x)\}, \quad \{x \in A : f(x) \geq g(x)\}, \quad \{x \in A : f(x) = g(x)\}$$
   are all measurable

3. We say a $\mathbb{C}$-valued function $f: A \to \mathbb{C}$ is measurable, if both $\mathrm{Re} f$ and $\mathrm{Im} f$ are measurable functions. Prove: $f: A \to \mathbb{C}$ is measurable if and only if for any open set $U \subset \mathbb{C} = \mathbb{R}^2$, $f^{-1}(U) \in \mathcal{L}$.

4. Let $f: A \to \mathbb{R}$ be a measurable function. Let $C \subset \mathbb{R}$ be a measurable subset. Is it true that $f^{-1}(C)$ is Lebesgue measurable?
   [Hint: Consider $\frac{x + \Phi(x)}{2}$, where Φ is the Cantor function.]
   [Not required to turn in: If $C \subset \mathbb{R}$ is a Borel set. Is it true that $f^{-1}(C)$ is Lebesgue measurable?]