Real Analysis Problem Set 4, Part 1

03/19/2018

Due 03/26, before class

1. Prove Borel-Cantelli lemma. [See Notes, page 2.]

2. Prove (1) If $f_n \to f$ in measure, $g_n \to g$ in measure, then $f_n + g_n \to f + g$ in measure.

   (2) If $f_n \to f$ in measure, $g_n \to g$ in measure and $m(A) < \infty$ then $f_n g_n \to f g$ in measure.

   (3) Show that the condition $m(A) < \infty$ can't be dropped via the example $f_n(x) = x$, $g_n(x) = \frac{1}{n}$.

3. We say $f_n \to f$ almost uniformly if $\forall \varepsilon > 0$, $\exists A_\varepsilon \in \mathcal{L}$ with $m(A_\varepsilon) < \varepsilon$, s.t. $f_n \to f$ uniformly on $A \setminus A_\varepsilon$.

   Prove: If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and $f_n \to f$ in measure.

4. Let f be a simple function defined on $A \in \mathcal{L}$.

   Prove: $\forall \varepsilon > 0$, $\exists$ closed set $F \subset A$ with $m(A \setminus F) < \varepsilon$ and continuous function g on $\mathbb{R}$ s.t. $f = g$ on F.