03/22/2018

Real Analysis Problem Set 4, Part 2

$\boxed{\text{Due 03/26, before class}}$

1. A real valued measurable function f is called a <u>semisimple function</u> if it has the form
$$f = \sum_{n=1}^{\infty} c_n \chi_{A_n}, \qquad \text{where } c_n \in \mathbb{R},\ A_n \in \mathcal{L} \text{ and } A_i \cap A_j = \phi.$$
Prove: Given any measurable function g, there exists a sequence of semisimple functions f_n such that $f_n \to g$ uniformly.

2. Let f be a measurable function.
Prove: $\exists$ a sequence of step functions $f_1, f_2, \cdots$ s.t. $f_n \to f$ a.e.

3. Read the proof of Tietze extension thm. (Page 52-53)
Think about the following question: Does the theorem/proof work for any metric space?

4. Given any measurable function f, define
$$\|f\|_{\mathcal{L}} = \inf\{r : m(\{x : |f(x)| > r\}) \le r\}$$
Prove: ① $m(\{x : |f(x)| > \|f\|_{\mathcal{L}}\}) \le \|f\|_{\mathcal{L}}$

② $\|f+g\|_{\mathcal{L}} \le \|f\|_{\mathcal{L}} + \|g\|_{\mathcal{L}}$

③ $f_n \to f$ in measure $\Leftrightarrow$ $\|f_n - f\|_{\mathcal{L}} \to 0$

④ For any $A \in \mathcal{L}$ and $c > 0$, $\|c\chi_A\|_{\mathcal{L}} = \inf\{c, m(A)\}$.