Real Analysis Problem Set 5, Part 1

03/26/2018

$\boxed{\text{Due 04/02, before class}}$

1. Suppose $\sum_{n=1}^{N} a_n \chi_{A_n} = \sum_{m=1}^{M} b_m \chi_{B_m}$, where $A_n, B_m \in \mathcal{L}$.

   Prove: $\sum_{n=1}^{N} a_n m(A_n) = \sum_{m=1}^{M} b_m m(B_m)$.

2. Assume f, g are non-negative and measurable.

   (1) If $m(A)=0$, Prove: For any function f, $\int_A f\,dx = 0$.

   (2) If $f=g$ a.e., Prove: $\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d} g(x)\,dx$.

3. (1) Construct a non-negative continuous function f on $\mathbb{R}$ s.t. f is integrable, but $\limsup_{x\to+\infty} f(x) = +\infty$.

   (2) Prove: If f is non-negative and uniformly continuous on $\mathbb{R}$, and f is integrable, then $\lim_{|x|\to+\infty} f(x) = 0$.

   ~~(3) Suppose f is non-negative and integrable~~

4. Let f, g be any non-negative function (which are not necessary measurable).

   Prove: (1) If $f \le g$ a.e., then $\underline{\int}_{\mathbb{R}^d} f(x)\,dx \le \underline{\int}_{\mathbb{R}^d} g(x)\,dx$, $\overline{\int}_{\mathbb{R}^d} f(x)\,dx \le \overline{\int}_{\mathbb{R}^d} g(x)\,dx$.

   (2) $\underline{\int}_{\mathbb{R}^d} (f+g)\,dx \ge \underline{\int}_{\mathbb{R}^d} f\,dx + \underline{\int}_{\mathbb{R}^d} g\,dx$.

   $\overline{\int}_{\mathbb{R}^d} (f+g)\,dx \le \overline{\int}_{\mathbb{R}^d} f\,dx + \overline{\int}_{\mathbb{R}^d} g\,dx$.

   (3) $\lim_{n\to\infty} \underline{\int}_{\mathbb{R}^d} \min(f(x), n)\,dx = \underline{\int}_{\mathbb{R}^d} f(x)\,dx$. [Hint: ε-trick for $\ge$].

   NOT required to turn in:

   (4) $\lim_{n\to\infty} \underline{\int}_{\mathbb{R}^d} f(x)\chi_{\{x:|x|\le n\}}(x)\,dx = \underline{\int}_{\mathbb{R}^d} f(x)\,dx$.

   (5) If f is measurable, bounded and $m(\operatorname{supp} f) < +\infty$, then

   $$\underline{\int}_{\mathbb{R}^d} f(x)\,dx = \overline{\int}_{\mathbb{R}^d} f(x)\,dx.$$

   (6) Use (2), (3), (4), (5) to show that if f, g are measurable, then $\int_{\mathbb{R}^d} (f+g)\,dx = \int_{\mathbb{R}^d} f\,dx + \int_{\mathbb{R}^d} g\,dx$.