Real Analysis Problem Set 5, Part 2

03/29/2018

Due 04/02, before class

1. (1) Suppose $f_1 \geq f_2 \geq \cdots \geq 0$ is a (decreasing) sequence of non-negative measurable functions, s.t. $\lim_{n\to\infty} f_n(x) = f$ a.e.
   Assume $\int_{\mathbb{R}^d} f_1(x)\,dx < +\infty$. Prove: $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$.

   (2) Let f_n be any sequence of non-negative measurable functions.
   Assume $\int_{\mathbb{R}^d} \sup_n f_n\,dx < +\infty$. Prove: $\limsup_{n\to\infty} \int_{\mathbb{R}^d} f_n\,dx \leq \int_{\mathbb{R}^d} \limsup_{n\to\infty} f_n\,dx$.

2. Suppose f_n is a sequence of non-negative measurable functions such that $f_n \to f$ a.e., and assume

$$\int_{\mathbb{R}^d} f_n(x)\,dx \longrightarrow \int_{\mathbb{R}^d} f(x)\,dx < +\infty$$

~~Prove~~ (1) Prove: $\forall A \in \mathcal{L}$, one has $\int_A f_n(x)\,dx \to \int_A f(x)\,dx$.

(2) Show that the conclusion of (1) could fail if $\int_{\mathbb{R}^d} f(x)\,dx = +\infty$.

3. Give a direct proof of Fatou's lemma (without using monotone convergence theorem), and then use Fatou's lemma to prove monotone convergence theorem.

4. Suppose f is non-negative and integrable in $[0,1]$. Moreover, suppose

$$\int_0^1 [f(x)]^n\,dx = \int_0^1 f(x)\,dx, \qquad n = 1, 2, 3, \ldots$$

Prove: $\exists$ measurable set $A \subset [0,1]$ s.t. $f = \chi_A$ a.e.

[Hint: Fatou]