Real Analysis Problem Set 6, Part 1

04/09/2018

Due 04/16, before class

1. (1) Define Lebesgue integrals for complex-valued functions $(\to L^1(\mathbb{R}^d, \mathbb{C}))$
[You need to justify "absolute integrability".]

(2) For complex-valued functions, <u>prove</u>

(a) $\int_{\mathbb{R}^d}(c_1 f_1 + c_2 f_2)\,dx = c_1\int_{\mathbb{R}^d} f_1\,dx + c_2\int_{\mathbb{R}^d} f_2\,dx$, $\forall c_1, c_2 \in \mathbb{C}$.

(b). $\int_{\mathbb{R}^d} \bar{f}\,dx = \overline{\int_{\mathbb{R}^d} f\,dx}$.

(c) $\left|\int_{\mathbb{R}^d} f(x)\,dx\right| \le \int_{\mathbb{R}^d} |f|\,dx$.

2. Let $f \in L^1(\mathbb{R}^d)$, and $\varepsilon > 0$. <u>Prove</u>:

(1) $\exists$ simple function $g \in L^1(\mathbb{R}^d)$ s.t. $\|f-g\|_{L^1(\mathbb{R}^d)} < \varepsilon$.

(2) $\exists$ step function $g \in L^1(\mathbb{R}^d)$ s.t. $\|f-g\|_{L^1(\mathbb{R}^d)} < \varepsilon$.

(3) $\exists$ continuous function $g \in L^1(\mathbb{R}^d)$ s.t. $\|f-g\|_{L^1(\mathbb{R}^d)} < \varepsilon$. [Moreover, one can make $\mathrm{supp}(g)$ compact.]

3. Let $f \in L^1(A)$.

(1) Suppose $A_1 \subset A_2 \subset \cdots \subset A$. <u>Prove</u>: $\int_{\bigcup_{n=1}^{\infty} A_n} f\,dx = \lim_{n\to\infty}\int_{A_n} f\,dx$

(2) Suppose $A \supset A_1 \supset A_2 \supset \cdots$. <u>Prove</u>: $\int_{\bigcap_{n=1}^{\infty} A_n} f\,dx = \lim_{n\to\infty}\int_{A_n} f\,dx$.

4. Let f be a non-negative measurable function. Denote

$$E_k = \{x : f(x) > 2^k\}, \quad F_k = \{x : 2^k < f(x) < 2^{k+1}\}, \quad k \in \mathbb{Z}.$$

(1). <u>Prove</u>: $f \in L^1(\mathbb{R}^d) \Leftrightarrow \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$
$\Leftrightarrow \sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$.

(2). Let $f(x) = \begin{cases} |x|^{-a}, & |x| \le 1 \\ 0, & |x| > 1, \end{cases}$ $\quad g(x) = \begin{cases} |x|^{-b}, & |x| > 1 \\ 0, & |x| \le 1. \end{cases}$

<u>Prove</u>: $f \in L^1(\mathbb{R}^d)$ if and only if $a < d$.
$g \in L^1(\mathbb{R}^d)$ if and only if $b > d$.