Real Analysis Problem Set 6, Part 2

04/12/2018

$\boxed{\text{Due 04/16, before class}}$

1. (A generalized LDCT).

Let $\{f_n\}$ be measurable functions on $A \in \mathcal{L}$, and $f_n \to f$ a.e. on A.
Suppose $\exists$ non-negative sequence $g_n \in L^1(A)$ s.t. $|f_n| \le g_n$ a.e. on A.
Moreover, suppose $g_n \to g$ a.e. on A and suppose $\lim_{n\to\infty} \int_A g_n \, dx = \int_A g \, dx < \infty$.
<u>Prove</u>: $f \in L^1(A)$, and $\|f_n - f\|_{L^1(A)} = \int_A |f_n(x) - f(x)| \, dx \to 0$.

[<u>Hint</u>: Modify the proof of LDCT.]

2. (Defect version of Fatou's lemma)

Suppose $f_n \in L^1(A)$ be non-negative, and $f_n \to f \in L^1(A)$ pointwise.
<u>Prove</u>: $\int_A f_n \, dx - \int_A f \, dx - \|f_n - f\|_{L^1(A)} \to 0$.

[<u>Hint</u>. Apply LDCT to $\min(f_n, f) = \frac{f_n + f}{2} - \frac{|f_n - f|}{2}$].

3. (Completeness of $L^p(A)$).

We can define the L^p $(p \ge 1)$ norm of a measurable function f on A to be
$$\|f\|_{L^p} := \left(\int_A |f|^p \, dx \right)^{1/p}$$
and define the L^p-space to be
$$L^p(A) := \{ f : \|f\|_{L^p} < \infty \}.$$
(1) Write down the exact meaning of "Cauchy sequence in $(L^p(A), \|\cdot\|_{L^p})$" and of "convergence in L^p-norm".

(2) State the theorem "$(L^p(A), \|\cdot\|_{L^p})$ is complete" and prove it.

[<u>Hint</u>: · $(a+b)^p \le 2^p(a^p + b^p)$ for $a, b > 0$ $\rightsquigarrow$ $(a_1 + \cdots + a_k)^p \le 2^p a_1^p + 2^{2p} a_2^p + \cdots + 2^{kp} a_k^p$.
· Choose sequence f_{n_k} s.t. $\|f_{n_k} - f_{n_{k-1}}\|_{L^p}^p \le 2^{-kp} \cdot 2^{-k}$.]

4. Construct a sequence of functions $f_n \in L^1(\mathbb{R})$ ~~s.t. $f_n \to f$ in L^1-norm, $f \in L^1$~~
s.t. (1) $f_n \to f$ in L^1-norm, where $f \in L^1(\mathbb{R})$
(2) For any $x \in \mathbb{R}$, $f_n(x) \not\to f(x)$.