Real Analysis Problem Set 7, Part 1

04/16/2018

Due 04/23, before class

1. Let $A \subset \mathbb{R}^d$ be measurable (and $m(A) < \infty$), and let f be a real-valued function defined on A.
   Let $F(x,y) := |f(x) - f(y)|$.
   Suppose $F \in L^1(A \times A)$. Prove: $f \in L^1(A)$.

2. [c.f. Page 2 of todays notes]
   Prove: $\overline{\Phi}(f) = \underline{\Phi}(f)$ a.e. $\Leftrightarrow f$ is continuous a.e.

3. (1) Suppose $f \in L^1([0,1] \times [0,1])$. Prove:
   $$\int_0^1 \left(\int_0^x f(x,y)\, dy \right) dx = \int_0^1 \left(\int_y^1 f(x,y)\, dx \right) dy$$
   (2) Suppose $f \in L^1((0,a))$. Let $g(x) = \int_x^a \frac{f(t)}{t}\, dt$. Prove: $g \in L^1((0,a))$, and
   $$\int_0^a g(x)\, dx = \int_0^a f(x)\, dx.$$

4. Prove: $\lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.
   [Hint: $\frac{1}{x} = \int_0^\infty e^{-tx}\, dt$.]