Real Analysis Problem Set 7, Part 2

04/19/2018

Due 04/23, before class

1. For $f \in L^1(\mathbb{R}^d)$, define $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx \quad (\xi \in \mathbb{R}^d)$.

   Prove: As $|\xi| \to +\infty$, $\hat{f}(\xi) \to 0$.

2. For $h > 0$, define $\eta_h(x) = h^{-\frac{d}{2}} e^{\frac{-\pi |x|^2}{h}}$, $x \in \mathbb{R}^d$.

   Prove: For any $f \in L^1(\mathbb{R}^d)$, $\| f * \eta_h - f \|_{L^1} \to 0$ as $h \to 0$.

   Note: We have "showed" that $f * \eta_h$ is smooth

3. Let $L: \mathbb{R}^d \to \mathbb{R}^d$ be a linear transform.

   Prove: For any measurable set $A \subset \mathbb{R}^d$, $m(L(A)) = |\det L| \cdot m(A)$.

   Hint: • One can write $L = L_1 \Delta L_2$, where L_1 is upper triangle matrix, with diagonal $= 1$
   L_2 is lower - - - - - - - - - -
   Δ is diagonal matrix.

   • For each case, first study the problem for $d=2$.
   e.g. if $L_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, then $\chi_{L(A)}(x_1, x_2) = \chi_A(L^{-1}(x_1, x_2)) = \chi_A(x - ay, y)$

4. Let f be a non-negative function on $\mathbb{R}^d$. Let
   $$\underline{G}(f) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \le y \le f(x)\} \subset \mathbb{R}^{d+1}$$

   Prove: (1) f is a measurable function if and only if $\underline{G}(f)$ is a measurable set.

   (2) Moreover, if (1) holds, then $\int_{\mathbb{R}^d} f(x)\, dx = m(\underline{G}(f))$.