Real Analysis Problem Set 8, Part 1

04/23/2018

Due 05/03, before class

1. Let $\varphi: X \to Y$ be a map between two sets.

(1) If $\mathcal{G}$ is a σ-algebra on Y. Prove: $\varphi^{-1}(\mathcal{G}) := \{\varphi^{-1}(B) \subset X : B \in \mathcal{G}\}$ is a σ-algebra on X.

(2) If $\mathcal{F}$ is a σ-algebra on X. Prove: $\varphi(\mathcal{F}) := \{B \subset Y : f^{-1}(B) \in \mathcal{F}\}$ is a σ-algebra on Y.

(3) For any family $\mathcal{K} \subset P(Y)$, Prove: $\langle \varphi^{-1}(\mathcal{K}) \rangle = \varphi^{-1}(\langle \mathcal{K} \rangle)$.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space, $A_n \in \mathcal{F}$. Prove:

(1) If $A_1 \subset A_2$, then $\mu(A_1) \leq \mu(A_2)$

(2) $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

(3) If $A_1 \subset A_2 \subset \cdots$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$

(4) If $A_1 \supset A_2 \supset \cdots$ and $\mu(A_1) < \infty$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

(5) (Fatou) $\mu(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mu(A_n)$

(6) (Fatou') If $\mu(\bigcup_{n=1}^{\infty} A_n) < +\infty$, then $\mu(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mu(A_n)$

(7) (Dominated convergence) If $A_n \to A$ (in the sense "$\chi_{A_n} \to \chi_A$ pointwise) and if $\exists F \in \mathcal{F}$ such that $\mu(F) < +\infty$ and $A_n \subset F$, $\forall n$.
Then $A \in \mathcal{F}$ and $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

3. Consider $X = S^{d-1} = \{(x_1, \cdots, x_d) \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 = 1\}$.
Let $\mathcal{F} = \{A \subset S^{d-1} : \text{the set } \{x \in \mathbb{R}^d : \frac{x}{|x|} \in A,\ 0<|x|<1\} \text{ is Lebesgue measurable in } \mathbb{R}^d\}$.
For any $A \in \mathcal{F}$, define $\mu(A) = d \cdot m(\{x \in \mathbb{R}^d : \frac{x}{|x|} \in A,\ 0<|x|<1\})$.
Prove: $(S^{d-1}, \mathcal{F}, \mu)$ is a measure space.

4. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. Let
$\mathcal{F} \otimes \mathcal{G}$ = the σ-algebra on $X \times Y$ generated by $\{A \times B : A \in \mathcal{F}, B \in \mathcal{G}\}$.
Then we get a product measurable space $(X \times Y, \mathcal{F} \otimes \mathcal{G})$.

Prove: (1) Suppose $E \in \mathcal{F} \otimes \mathcal{G}$. Then for any $x \in X$ and $y \in Y$, one has
$E_x := \{y \in Y : (x, y) \in E\} \in \mathcal{G}$, $\quad E^y := \{x \in X : (x, y) \in E\} \in \mathcal{F}$.
[Hint: Show that $\mathcal{H} := \{E \subset X \times Y : E_x \in \mathcal{G}, E^y \in \mathcal{F}, \forall x, y\}$ is a σ-algebra]

(2) $\mathcal{B}(\mathbb{R}^{d_1 + d_2}) = \mathcal{B}(\mathbb{R}^{d_1}) \otimes \mathcal{B}(\mathbb{R}^{d_2})$

(the Borel algebra)