Real Analysis Problem Set 8, Part 2

04/26/2018

Due 05/03, before class

1. Let $(X, \mathcal{F})$ be a measurable space. Prove:
   (1) $A \in \mathcal{F} \Leftrightarrow \chi_A$ is measurable
   (2) If f, g are $\mathbb{R}$-valued (or $[0, +\infty]$-valued) measurable functions, so are cf, $f+g$, fg
   (3) If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, $f : X \to \mathbb{R}$ is measurable, then $\varphi \circ f : X \to \mathbb{R}$ is measurable
   (4) If f_n are measurable functions, then $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$, $\sup_n f_n$, $\inf_n f_n$ are measurable.

2. Prove the monotone convergence theorem [Page 3 on the notes.]

3. Prove the Borel-Cantelli lemma: Let $A_1, A_2, \cdots$ be measurable sets in a measure space $(X, \mathcal{F}, \mu)$, such that

$$\sum_{n=1}^{\infty} \mu(A_n) < +\infty.$$

Then almost every $x \in X$ is contained in at most finitely many of the E_n's.

Then give a counterexample that shows the conclusion could fail if one replace the condition $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$ by $\lim_{n\to\infty} \mu(A_n) = 0$.

4. Let $(X, \mathcal{F})$, $(Y, \mathcal{G})$ be measurable spaces, and $\varphi : X \to Y$ a measurable map. Suppose μ is a measure on $(X, \mathcal{F})$. We define a "push-forward measure" $\varphi_* \mu$ on $(Y, \mathcal{G})$ by

$$(\varphi_* \mu)(B) := \mu(\varphi^{-1}(B)), \quad \forall B \in \mathcal{G}.$$

   (1) Prove: $\varphi_* \mu$ is a measure on $(Y, \mathcal{G})$.
   (2) Let $f : Y \to \mathbb{R}$ be either a non-negative measurable function or an integrable function. Prove:

$$\int_X f \circ \varphi \, d\mu = \int_Y f \, d\varphi_* \mu.$$